3D rotation sequences (e.g., yaw/pitch/roll) versus 3D rotations via quaternions

Prof Thomas Bewley
UC San Diego & USAF Academy

The **configuration** of a solid body is defined by **6 DOF** (degrees of freedom):
Location $\{x, y, z\}$ is defined by **3 DOF** (vector $\vec{r}$ from reference frame origin to CM).
Orientation is defined by additional **3 DOF**, and is much more difficult to describe….

Start by defining a **reference orientation** of the principle axes:
$\{\vec{b}^1, \vec{b}^2, \vec{b}^3\} = \{N, E, D\}$ used by aerospace engineers & SAE standard
$\{\vec{b}^1, \vec{b}^2, \vec{b}^3\} = \{E, N, U\}$ used by ISO standard & meteorologists ("zed in the clouds")

At any moment, the vehicle **orientation** is defined as one finite **rotation** (described by a rotation matrix R) from a chosen **reference orientation** (NED or ENU), **not** as the set of many small rotations that the vehicle actually underwent to ultimately get pointed that way.

Complex variables

$i^2 = -1$ Euler 1748 $\qquad z = a + bi = Re^{i\phi} = R(\cos\phi + i\sin\phi)$

- Math: $(a + bi) + (c + di) = (a + c) + (b + d)i, \quad (c + si)(a + bi) = (ca - sb) + (sa + cb)\,i$
- **2D rotation** & scaling: $z_1 = R_1 e^{i\phi_1},\ z_2 = R_2 e^{i\phi_2} \ \Rightarrow\ z_3 = z_1 z_2 = (R_1 R_2)\,e^{i(\phi_1+\phi_2)} = R_3\,e^{i\phi_3}$
- **Fundamental Thm of Algebra**: $s^n + a_{n-1}s^{n-1} + \ldots + a_0 = (s - s_1)(s - s_2)\ \cdots\ (s - s_n)$

  Argand 1806 — roots are complex!

Quaternions

$i^2 = j^2 = k^2 = ijk = -1$ Hamilton 1843

$\mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k = R\,e^{\vec{u}\,\phi} = R(\cos\phi + \vec{u}\sin\phi)$
where $\vec{u} = u_1 i + u_2 j + u_3 k$ with $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 = 1$

- Math (noncommutative!): $\mathbf{p}\,\mathbf{q} = \ldots$ with $ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$
- Inverse: $\mathbf{p}^* = p_0 - p_1 i - p_2 j - p_3 k,\quad \|\mathbf{p}\| = (p_0^2 + p_1^2 + p_2^2 + p_3^2)^{1/2},\quad \mathbf{p}^{-1} = \mathbf{p}^*/\|\mathbf{p}\|^2$
- **3D rotation**: $\overrightarrow{w}' = \mathbf{p}\,\overrightarrow{w}\,\mathbf{p}^*$ with $\|\mathbf{p}\| = 1$ rotates $\overrightarrow{w}$ an angle 2ϕ about the unit vector $\vec{u}$

Quaternion Math (noncommutative!)

$i^2 = j^2 = k^2 = ijk = -1$

$\mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k = R\,e^{\vec{u}\,\phi} = R(\cos\phi + \vec{u}\sin\phi)$
where $\vec{u} = u_1 i + u_2 j + u_3 k$ with $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 = 1$

$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$

$\mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k,\ \mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k,\quad \mathbf{r} = \mathbf{p}\,\mathbf{q}$

$$\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

Denoting $\mathbf{p} = p_0 + \vec{p}$ and $\mathbf{q} = q_0 + \vec{q}$ where, e.g., $\vec{p} = p_1 i + p_2 j + p_3 k$ (p_0 is "real part", $\vec{p}$ is "vec part"), we may also write

$$\mathbf{p}\,\mathbf{q} = (p_0 + \vec{p})(q_0 + \vec{q}) = (p_0\,q_0 - \vec{p}\cdot\vec{q}) + (p_0\,\vec{q} + q_0\,\vec{p} + \vec{p}\times\vec{q})$$

Quaternion Math (noncommutative!)

$i^2 = j^2 = k^2 = ijk = -1$

$\mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k = R\, e^{\vec{u}\,\phi} = R(\cos\phi + \vec{u}\sin\phi)$

where $\vec{u} = u_1 i + u_2 j + u_3 k$ with $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 = 1$

$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$

conjugate of $\mathbf{q} = q_0 + \vec{q}$ is $\mathbf{q}^* = q_0 - q_1 i - q_2 j - q_3 k = q_0 - \vec{q}$

then: $(\mathbf{q}\,\mathbf{p})^* = \mathbf{p}^*\,\mathbf{q}^*,\ q_0 = (\mathbf{q} + \mathbf{q}^*)/2,\ \vec{q} = (\mathbf{q} - \mathbf{q}^*)/2$

norm of $\mathbf{q}$ is $\|\mathbf{q}\| = \sqrt{\mathbf{q}\,\mathbf{q}^*} = \sqrt{\mathbf{q}^*\,\mathbf{q}} = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}$

A unit quaternion is a quaternion $\mathbf{q}$ such that $\|\mathbf{q}\| = 1$, and may always be written $\mathbf{q} = e^{\vec{u}\,\phi}$ with $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 = 1$.

inverse of $\mathbf{q}$ given by $\mathbf{q}^{-1} = \mathbf{q}^*/\|\mathbf{q}\|^2 \quad \Rightarrow \quad \mathbf{q}\,\mathbf{q}^{-1} = \mathbf{q}^{-1}\,\mathbf{q} = 1$

Recall from last time:

Rodrigues' Rotation Formula: *Define $\vec{w}\,'$ as the right-handed rotation of $\vec{w}$ about a unit vector $\vec{u}$ by an angle θ. Then*

$$\vec{w}\,' = \vec{w}\cos\theta + (\vec{u}\cdot\vec{w})\,\vec{u}\,(1-\cos\theta) + (\vec{u}\times\vec{w})\ \sin\theta$$

Proof: Decompose $\vec{w} = \vec{w}_{\|} + \vec{w}_{\perp}$, components parallel and perpendicular to $\vec{u}$ .... □

Some useful identities

§A of *RR*

$$e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x \quad \Rightarrow \quad e^{\mathrm{i}\pi} + 1 = 0 \qquad \text{(B.44)}$$

$$e^{\mathrm{i}x}e^{-\mathrm{i}x} = 1 \quad \Rightarrow \quad \cos^2 x + \sin^2 x = 1 \qquad \text{(B.45)}$$

$$\sin x = (e^{\mathrm{i}x} - e^{-\mathrm{i}x})/(2\mathrm{i}) \qquad \text{(B.46)}$$

$$\cos x = (e^{\mathrm{i}x} + e^{-\mathrm{i}x})/2 \qquad \text{(B.47)}$$

$$\sinh x = (e^{x} - e^{-x})/2 \qquad \text{(B.48)}$$

$$\cosh x = (e^{x} + e^{-x})/2 \qquad \text{(B.49)}$$

$$\tan x = \sin(x)/\cos(x) \qquad \text{(B.50)}$$

$$\tanh x = \sinh(x)/\cosh(x) \qquad \text{(B.51)}$$

$$\sin(x+y) = \sin x \, \cos y + \cos x \, \sin y \qquad \text{(B.52)}$$

$$\cos(x+y) = \cos x \, \cos y - \sin x \, \sin y \qquad \text{(B.53)}$$

$$\cos x = 2\cos^2(x/2) - 1 = 1 - 2\sin^2(x/2) = \cos^2(x/2) - \sin^2(x/2) \qquad \text{(B.54)}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \, \tan y} \qquad \text{(B.55)}$$

$$2\,\sin(x)\,\cos(y) = \sin(x+y) + \sin(x-y) \qquad \text{(B.56)}$$

$$2\,\cos(x)\,\cos(y) = \cos(x+y) + \cos(x-y) \qquad \text{(B.57)}$$

$$2\,\sin(x)\,\sin(y) = \cos(x-y) - \cos(x+y) \qquad \text{(B.58)}$$

$$\sum_{m=1}^{M} \cos mx = \frac{\cos[(M+1)x/2]\sin(Mx/2)}{\sin(x/2)} \qquad \text{(B.59)}$$

$$a\,\sin x + b\,\cos x = r\,\sin(x+\alpha) \qquad \text{(B.60)}$$

$$a\,\cos x + b\,\sin x = r\,\cos(x-\alpha) \qquad \text{(B.61)}$$

$$\operatorname{acos} x = \frac{\pi}{2} - \operatorname{asin} x \qquad \text{(B.62)}$$

$$d(\sin x)/dx = \cos x; \quad d(\cos x)/dx = -\sin x \qquad \text{(B.63)}$$

$$d(\tan x)/dx = 1/\cos^2 x \qquad \text{(B.64)}$$

$$d(\sinh x)/dx = \cosh x; \quad d(\cosh x)/dx = \sinh x \qquad \text{(B.65)}$$

$$d(\tanh x)/dx = 1 - \tanh^2 x = 1/\cosh^2(x) = \operatorname{sech}^2(x) \qquad \text{(B.66)}$$

$$d(\operatorname{asin} x)/dx = 1/\sqrt{1-x^2} \qquad \text{(B.67)}$$

$$d(\operatorname{acos} x)/dx = -1/\sqrt{1-x^2} \qquad \text{(B.68)}$$

$$d(\operatorname{atan} x)/dx = 1/(1+x^2) \qquad \text{(B.69)}$$

$$d(\ln x)/dx = 1/x \qquad \text{(B.70)}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \triangleq T_n \quad \text{Triangular numbers} \qquad \text{(B.71)}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{(B.72)}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \qquad \text{(B.73)}$$

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \qquad \text{(B.74)}$$

$$\sum_{k=1}^{n} T_k = \frac{n(n+1)(n+2)}{6} \triangleq Te_n \quad \text{Tetrahedral numbers} \qquad \text{(B.75)}$$

$$\sum_{k=1}^{n} Te_k = \frac{n(n+1)(n+2)(n+3)}{24} \triangleq P_n \quad \text{Pentatope numbers} \qquad \text{(B.76)}$$

Rotation using Quaternions

$i^2 = j^2 = k^2 = ijk = -1$ $\quad$ $\mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k = R\,e^{\vec{u}\,\phi} = R(\cos\phi + \vec{u}\sin\phi)$

where $\vec{u} = u_1 i + u_2 j + u_3 k$ with $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 = 1$

$\overrightarrow{w}' = \mathbf{p}\,\overrightarrow{w}\,\mathbf{p}^*$ with $\|\mathbf{p}\| = 1$ ($R = 1$ above) rotates $\overrightarrow{w}$ by $\theta = 2\phi$ about the unit vector $\vec{u}$

Proof: Recall that $\mathbf{p}\,\mathbf{q} = (p_0 + \vec{p})(q_0 + \vec{q}) = (p_0 q_0 - \vec{p}\cdot\vec{q}) + (p_0\,\vec{q} + q_0\,\vec{p} + \vec{p}\times\vec{q})$. Then:

$$\overrightarrow{w}' = \mathbf{p}\,\mathbf{w}\,\mathbf{p}^* = \big(\cos\phi + \mathbf{u}\,\sin\phi\big)\,\mathbf{w}\,\big(\cos\phi - \mathbf{u}\,\sin\phi\big)$$

$$= \mathbf{w}\,\cos^2\phi + (\mathbf{u}\,\mathbf{w} - \mathbf{w}\,\mathbf{u})\,\sin\phi\,\cos\phi - \mathbf{u}\,\mathbf{w}\,\mathbf{u}\,\sin^2\phi$$

$$= \overrightarrow{w}\,\cos^2\phi + 2\,\vec{u}\times\overrightarrow{w}\,\sin\phi\,\cos\phi + \mathbf{u}\,[\overrightarrow{w}\cdot\vec{u} - \overrightarrow{w}\times\vec{u}]\,\sin^2\phi$$

$$= \overrightarrow{w}\,\cos^2(\theta/2) + \vec{u}\times\overrightarrow{w}\,\sin\theta + [\vec{u}\,(\overrightarrow{w}\cdot\vec{u}) + \vec{u}\cdot(\overrightarrow{w}\times\vec{u}) - \vec{u}\times(\overrightarrow{w}\times\vec{u})]\,\sin^2(\theta/2)$$

Some useful identities

§A of *RR*

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\,\mathbf{e}_1 + (a_3b_1 - a_1b_3)\,\mathbf{e}_2 + (a_1b_2 - a_2b_1)\,\mathbf{e}_3 \qquad \text{(B.17)}$$

$$\vec{a}\times\vec{b} = [\vec{a}]_\times\,\vec{b}, \qquad [\vec{a}]_\times \triangleq \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) \qquad \text{(B.20)}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) \qquad \text{(B.21)}$$

$$\|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\|^2\,\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2 \qquad \text{(B.22)}$$

Rotation using Quaternions

$i^2 = j^2 = k^2 = ijk = -1$ $\quad$ $\mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k = R\,e^{\vec{u}\,\phi} = R(\cos\phi + \vec{u}\sin\phi)$

where $\vec{u} = u_1 i + u_2 j + u_3 k$ with $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 = 1$

$\overrightarrow{w}' = \mathbf{p}\,\overrightarrow{w}\,\mathbf{p}^*$ with $\|\mathbf{p}\| = 1$ ($R = 1$ above) rotates $\overrightarrow{w}$ by $\theta = 2\phi$ about the unit vector $\vec{u}$

Proof: Recall that $\mathbf{p}\,\mathbf{q} = (p_0 + \vec{p})(q_0 + \vec{q}) = (p_0\,q_0 - \vec{p}\cdot\vec{q}) + (p_0\,\vec{q} + q_0\,\vec{p} + \vec{p}\times\vec{q})$. Then:

$$\begin{aligned}
\overrightarrow{w}' = \mathbf{p}\,\mathbf{w}\,\mathbf{p}^* &= \left(\cos\phi + \mathbf{u}\,\sin\phi\right)\mathbf{w}\left(\cos\phi - \mathbf{u}\,\sin\phi\right) \\
&= \mathbf{w}\,\cos^2\phi + (\mathbf{u}\,\mathbf{w} - \mathbf{w}\,\mathbf{u})\,\sin\phi\,\cos\phi - \mathbf{u}\,\mathbf{w}\,\mathbf{u}\sin^2\phi \\
&= \overrightarrow{w}\,\cos^2\phi + 2\,\vec{u}\times\overrightarrow{w}\,\sin\phi\,\cos\phi + \mathbf{u}\,[\overrightarrow{w}\cdot\vec{u} - \overrightarrow{w}\times\vec{u}]\,\sin^2\phi \\
&= \overrightarrow{w}\,\cos^2(\theta/2) + \vec{u}\times\overrightarrow{w}\,\sin\theta + [\vec{u}\,(\overrightarrow{w}\cdot\vec{u}) + \vec{u}\cdot(\overrightarrow{w}\times\vec{u}) - \vec{u}\times(\overrightarrow{w}\times\vec{u})]\,\sin^2(\theta/2) \\
&= \overrightarrow{w}\,\cos^2(\theta/2) + \vec{u}\times\overrightarrow{w}\,\sin\theta + [\vec{u}\,(\overrightarrow{w}\cdot\vec{u}) + 0 - \overrightarrow{w}\,(\vec{u}\cdot\vec{u}) + \vec{u}\,(\vec{u}\cdot\overrightarrow{w})]\,\sin^2(\theta/2) \\
&= \overrightarrow{w}\left[\cos^2(\theta/2) - \sin^2(\theta/2)\right] + \vec{u}\times\overrightarrow{w}\,\sin\theta + 2\,\vec{u}\,(\vec{u}\cdot\overrightarrow{w})\,\sin^2(\theta/2) \\
&= \overrightarrow{w}\,\cos\theta + (\vec{u}\cdot\overrightarrow{w})\,\vec{u}\,(1 - \cos\theta) + \vec{u}\times\overrightarrow{w}\,\sin\theta
\end{aligned}$$

← **Rodrigues' Rotation Formula** $\square$

Complex variables

$i^2 = -1$ Euler 1748 $\quad z = a + bi = Re^{i\phi} = R(\cos\phi + i\sin\phi)$

- Math: $(a+bi)+(c+di) = (a+c)+(b+d)i, \quad (c+si)(a+bi) = (ca-sb)+(sa+cb)\,i$
- **2D rotation** & scaling: $z_1 = R_1 e^{i\phi_1},\ z_2 = R_2 e^{i\phi_2} \Rightarrow z_3 = z_1 z_2 = (R_1R_2)\,e^{i(\phi_1+\phi_2)} = R_3\,e^{i\phi_3}$
- **Fundamental Thm of Algebra**: $s^n + a_{n-1}s^{n-1} + \ldots + a_0 = (s-s_1)(s-s_2)\cdots(s-s_n)$ ← roots are complex!
  Argand 1806

Quaternions

$i^2 = j^2 = k^2 = ijk = -1$ Hamilton 1843 $\quad \mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k = R\,e^{\vec{u}\,\phi} = R(\cos\phi + \vec{u}\sin\phi)$

where $\vec{u} = u_1 i + u_2 j + u_3 k$ with $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 = 1$

- Math (noncommutative!): $\mathbf{p\,q} = \ldots$ with $ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$
- Inverse: $\mathbf{p}^* = p_0 - p_1 i - p_2 j - p_3 k, \quad \|\mathbf{p}\| = (p_0^2+p_1^2+p_2^2+p_3^2)^{1/2}, \quad \mathbf{p}^{-1} = \mathbf{p}^*/\|\mathbf{p}\|^2$
- **3D rotation**: $\vec{w}\,' = \mathbf{p}\,\vec{w}\,\mathbf{p}^*$ with $\|\mathbf{p}\| = 1$ rotates $\vec{w}$ an angle 2ϕ about the unit vector $\vec{u}$

Quaternions: wrap up

$i^2 = j^2 = k^2 = ijk = -1$ Hamilton 1843 $\quad \mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k = R\,e^{\vec{u}\,\phi} = R(\cos\phi + \vec{u}\sin\phi)$

where $\vec{u} = u_1 i + u_2 j + u_3 k$ with $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 = 1$

- Math (noncommutative!): $\mathbf{p\,q} = \ldots$ with $ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$
- Inverse: $\mathbf{p}^* = p_0 - p_1 i - p_2 j - p_3 k, \quad \|\mathbf{p}\| = (p_0^2+p_1^2+p_2^2+p_3^2)^{1/2}, \quad \mathbf{p}^{-1} = \mathbf{p}^*/\|\mathbf{p}\|^2$
- **3D rotation**: $\vec{w}\,' = \mathbf{p}\,\vec{w}\,\mathbf{p}^*$ with $\|\mathbf{p}\| = 1$ rotates $\vec{w}$ an angle 2ϕ about the unit vector $\vec{u}$

Looking at $\vec{w} = \vec{e}^{\,1}$, $\vec{w} = \vec{e}^{\,2}$, and $\vec{w} = \vec{e}^{\,3}$ separately, one may easily determine that:

$$\begin{pmatrix} w_1' \\ w_2' \\ w_3' \end{pmatrix} = R_{\mathbf{p}} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \quad \text{with} \quad R_{\mathbf{p}} = \begin{pmatrix} p_0^2+p_1^2-p_2^2-p_3^2 & 2p_1p_2-2p_0p_3 & 2p_1p_3+2p_0p_2 \\ 2p_1p_2+2p_0p_3 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_2p_3-2p_0p_1 \\ 2p_1p_3-2p_0p_2 & 2p_2p_3+2p_0p_1 & p_0^2-p_1^2-p_2^2+p_3^2 \end{pmatrix}$$

Composition of rotations: if $\|\mathbf{q}\| = 1$ and $\|\mathbf{r}\| = 1$ and $\mathbf{s} = \mathbf{r\,q}$ then $\|\mathbf{s}\| = 1$.

If $\vec{w}\,' = \mathbf{q}\,\vec{w}\,\mathbf{q}^*$ and $\vec{w}\,'' = \mathbf{r}\,\vec{w}\,'\,\mathbf{r}^*$ then $\vec{w}\,'' = \mathbf{s}\,\vec{w}\,\mathbf{s}^*$ where $\mathbf{s} = \mathbf{r\,q}$.

Euler and Tait-Bryan Rotation sequences

3D rotations can also be defined by **three successive rotations**, of angles $\{\alpha, \beta, \gamma\}$, about the **body coordinate axes.** These three rotations can be selected and ordered in twelve different ways, which fall into two general categories:

1) about each of the three different coordinate axis, that is, via one of the following 6 choices: **{1,2,3}, {1,3,2}, {2,1,3}, {2,3,1}, {3,1,2}, {3,2,1}**, called **Tait-Bryan** rotation sequences, or

2) by taking the 3rd rotation axis same as the 1st, resulting in one of the following 6 choices: **{1,2,1}, {1,3,1}, {2,1,2}, {2,3,2}, {3,1,3}, {3,2,3}**, called **Euler** rotation sequences.

We will illustrate via example the {3,2,1} and {3,1,3} rotation sequences below, and compare them to rotation via the use of quaternions; other examples, using these and the other 10 rotation sequences, are natural generalizations.

Examples of solid body rotations:	Roll by $\pi/2$, then pitch by $\pi/2$	Pitch by $\pi/2$, then yaw by $\pi/2$
3-2-1 Tait-Bryan (intrinsic) $\{\alpha, \beta, \gamma\} =$ {yaw, pitch, roll}	$\{0,0,0\} \underset{\gamma\nearrow}{\longrightarrow} \left\{0,0,\frac{\pi}{2}\right\} \underset{\alpha\nearrow}{\longrightarrow} \left\{\frac{\pi}{2},0,\frac{\pi}{2}\right\}$	$\{0,0,0\} \underset{\beta\nearrow}{\longrightarrow} \left\{0,\frac{\pi}{2},0\right\}$ $\Updownarrow$ equivalent (singularity!) $\left\{\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right\} \underset{\beta\searrow}{\longrightarrow} \left\{\frac{\pi}{2},0,\frac{\pi}{2}\right\}$
3-1-3 Euler (intrinsic) $\{\alpha, \beta, \gamma\} =$ {yaw, roll, yaw}	$\{0,0,0\} \underset{\beta\nearrow}{\longrightarrow} \left\{0,\frac{\pi}{2},0\right\} \underset{\alpha\nearrow}{\longrightarrow} \left\{\frac{\pi}{2},\frac{\pi}{2},0\right\}$	$\{0,0,0\}$ $\Updownarrow$ equivalent (singularity!) $\left\{\frac{\pi}{2},0,-\frac{\pi}{2}\right\} \underset{\beta\nearrow}{\longrightarrow} \left\{\frac{\pi}{2},\frac{\pi}{2},-\frac{\pi}{2}\right\} \underset{\gamma\nearrow}{\longrightarrow} \left\{\frac{\pi}{2},\frac{\pi}{2},0\right\}$
quaternions (extrinsic)	$\vec{u}_1 = i,\ \ \theta_1 = \pi/2; \quad \vec{u}_2 = k,\ \ \theta_1 = \pi/2;$ $\mathbf{q}_1 = e^{\vec{u}_1\phi_1} = \cos(\theta_1/2) + \vec{u}_1 \sin(\theta_1/2) = \left(\sqrt{2}/2\right)(1+i)$ $\mathbf{q}_2 = e^{\vec{u}_2\phi_2} = \cos(\theta_2/2) + \vec{u}_2 \sin(\theta_2/2) = \left(\sqrt{2}/2\right)(1+k)$ $\mathbf{q} = \mathbf{q}_2\,\mathbf{q}_1 = (1/2)\,(1+k)\,(1+i) = (1/2)\,(1+i+k+ki)$ $= \cos\frac{\pi}{3} + \frac{i+j+k}{\sqrt{3}}\sin\frac{\pi}{3} \ \Rightarrow\ \theta = \frac{2\pi}{3},\ \vec{u} = \frac{i+j+k}{\sqrt{3}}$	$\vec{u}_1 = j,\ \ \theta_1 = \pi/2; \quad \vec{u}_2 = i,\ \ \theta_1 = \pi/2;$ $\mathbf{q}_1 = e^{\vec{u}_1\phi_1} = \cos(\theta_1/2) + \vec{u}_1 \sin(\theta_1/2) = \left(\sqrt{2}/2\right)(1+j)$ $\mathbf{q}_2 = e^{\vec{u}_2\phi_2} = \cos(\theta_2/2) + \vec{u}_2 \sin(\theta_2/2) = \left(\sqrt{2}/2\right)(1+i)$ $\mathbf{q} = \mathbf{q}_2\,\mathbf{q}_1 = (1/2)\,(1+i)\,(1+j) = (1/2)\,(1+i+k+ij)$ $= \cos\frac{\pi}{3} + \frac{i+j+k}{\sqrt{3}}\sin\frac{\pi}{3} \ \Rightarrow\ \theta = \frac{2\pi}{3},\ \vec{u} = \frac{i+j+k}{\sqrt{3}}$

Rotation matrices corresponding to the Euler and Tait-Bryan Rotation sequences

Recall that the **{3,2,1} Tait-Bryan** rotation sequence is defined by three successive rotations (yaw by α, then pitch by β, then roll by γ), and thus may be written as a rotation matrix as

$$R_{321} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix} \begin{pmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 c_2 & c_2 s_1 & -s_2 \\ c_1 s_2 s_3 - c_3 s_1 & c_1 c_3 + s_1 s_2 s_3 & c_2 s_3 \\ s_1 s_3 + c_1 c_3 s_2 & c_3 s_1 s_2 - c_1 s_3 & c_2 c_3 \end{pmatrix}$$

with $c_1 = \cos\alpha$, $s_1 = \sin\alpha$, $c_2 = \cos\beta$, $s_2 = \sin\beta$, $c_3 = \cos\gamma$, $s_3 = \sin\gamma$.

Similarly, the **{3,1,3} Euler** rotation sequence is also defined by three successive rotations. (yaw by α, then roll by β, then yaw again by γ), and thus may be written as a rotation matrix as

$$R_{313} = \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 c_3 - c_2 s_1 s_3 & c_3 s_1 + c_1 c_2 s_3 & s_2 s_3 \\ -c_1 s_3 - c_2 c_3 s_1 & c_1 c_2 c_3 - s_1 s_3 & c_3 s_2 \\ s_1 s_2 & -c_1 s_2 & c_2 \end{pmatrix}$$

Summary: 3D Rotations

Any 3D rotation may be achieved by performing a single rotation of angle θ about a single, carefully-selected 3D unit vector $\vec{u}$, either

a) using **Rodrigues' rotation formula** directly, or

b) representing this formula as a **rotation matrix** R, or

c) leveraging this formula implemented as a **unit quaternion**, which is the most convenient.

This approach has 3 DOF: the {latitude, longitude} defining $\vec{u}$, and the angle θ. Successive rotations may be accounted for simply by multiplying the corresponding quaternions.
Note: in quaternion math, **order matters.** Quaternion approach is **singularity-free** :) but is a **double cover**: if a quaternion $\mathbf{q}$ describes a rotation, then $-\mathbf{q}$ describes the same rotation. :(

3D rotations can also be defined by **three successive rotations**, of angles $\{\alpha, \beta, \gamma\}$, about the **body coordinate axes**; e.g., the **{3,2,1}={yaw, pitch, roll}** **Tait-Bryan rotation sequence** common in aerodynamics, and the **{3,1,3}={yaw, roll, yaw again}** **Euler rotation sequence** common in physics, were illustrated in this talk. All such rotation sequences are **singular**. :(