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A fundamental limit on the balance of power in a transpiration-controlled channel flow

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This paper is a direct sequel to Bewley & Aamo (*J. Fluid Mech.,* vol. 499, 2004, pp. 183–196). It was conjectured in that paper, based on the numerical evidence available at that time, that the minimum drag of a constant mass flux channel flow might in fact be that of the laminar flow. This conjecture turned out to be false; Min *et al.* (*J. Fluid Mech.*, vol. 558, 2006, 309318) discovered a curious control strategy which in fact reduces the time-averaged drag to sub-laminar levels. The present paper establishes rigorously that the power of the control input applied at the walls is always larger than the power saved (due to drag reduction below the laminar level) for any possible control distribution, including that proposed by Min *et al.* (2006), thus establishing that, energetically (that is accounting for the power saved due to drag reduction and the power exerted by application of the control), the optimal control solution is necessarily to relaminarize the flow.

1. Main result

Following Bewley & Aamo (2004), consider a constant mass flux three-dimensional channel flow governed by the incompressible Navier–Stokes equation

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p &= \nu\Delta\boldsymbol{u} + \boldsymbol{i}P_x, \\ \nabla\cdot\boldsymbol{u} &= 0, \end{aligned} \tag{1.1}$$

governing the flow in the rectangular domain Ω of size $(0, L_x) \times (-1, 1) \times (0, L_z)$, as shown in figure 1, with x_1 the streamwise direction, x_2 the wall-normal direction and x_3 the spanwise direction, with corresponding velocity components u_1, u_2, u_3. The mean pressure gradient $P_x(t)$ in the streamwise direction $\boldsymbol{i}$ is adjusted in such a way as to maintain a constant bulk velocity

$$U_B = \frac{1}{V}\int_\Omega u_1(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = \text{constant} \qquad \forall t, \tag{1.2}$$

where $V = 2L_xL_z$. Define $U(x_2) = C(1 - x_2^2)$, where $C = (3/2)U_B$, as the streamwise component of the laminar (parabolic) velocity profile with the same bulk velocity, and define the fluctuating component of the velocity

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{i}U(x_2). \tag{1.3}$$

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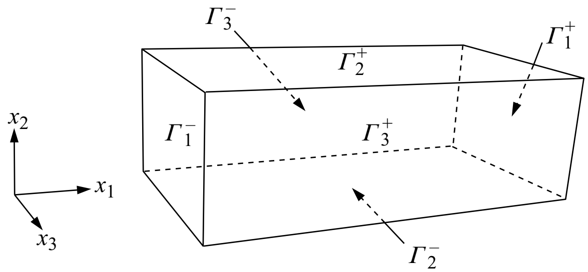

FIGURE 1. Flow domain Ω considered and the notation used.

Define the 'infinite time-average drag' (denoted $\langle D\rangle_\infty$) as the time average (denoted $\langle D\rangle_T$) of the instantaneous drag (denoted $D(t)$) as the averaging time T approaches infinity, i.e.

$$\langle D\rangle_\infty \triangleq \lim_{T\to\infty}\langle D\rangle_T \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t \triangleq \lim_{T\to\infty}\frac{-\mu}{T}\int_0^T\int_{\Gamma_2^\pm}\frac{\partial u_1}{\partial n}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

where $\boldsymbol{n}$ is an outward facing normal; $\Gamma_2^\pm$ denotes the set of points given by the union of the upper and lower walls of the channel; $\rho=1$ is the density (assuming the system is appropriately normalized); and $\mu=\nu$ is the viscosity. Let D_L denote the drag of the corresponding laminar channel flow with the same dimensions, viscosity and bulk velocity. Define also $\phi(\boldsymbol{x},t)=-\boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{n}$ as the unsteady blowing/suction distribution applied to the walls $\Gamma_2^\pm$ of the channel flow system as the control (i.e. $\phi>0$ corresponds to blowing into the channel, and $\phi<0$ corresponds to suction from the channel). We also define the instantaneous energy dissipation rate as

$$\nu\,\|\nabla\boldsymbol{u}\|_2^2=\nu\sum_{\iota,\kappa=1}^{3}\left\|\frac{\partial u_\iota}{\partial x_\kappa}\right\|_2^2.$$

With these definitions, it is established in (4.7), (4.9), (4.12) and (4.13) of Bewley & Aamo (2004) that

$$\langle D\rangle_\infty U_B V=\left\langle\nu\,\|\nabla\boldsymbol{u}\|_2^2\right\rangle_\infty-\left\langle\int_{\Gamma_2^\pm}\phi\,(p+\phi^2/2)\,\mathrm{d}\boldsymbol{x}\right\rangle_\infty, \tag{1.4}$$

$$D_L U_B V=\nu L_x L_z\int_{-1}^{1}U'^2\mathrm{d}\boldsymbol{x}, \tag{1.5}$$

$$\|\nabla\boldsymbol{u}\|_2^2=L_xL_z\int_{-1}^{1}U'^2\,\mathrm{d}x_2+\underbrace{2\int_\Omega U'\frac{\partial v_1}{\partial x_2}\mathrm{d}\boldsymbol{x}}_{=0}+\|\nabla\boldsymbol{v}\|_2^2, \tag{1.6}$$

noting that

(i) relation (1.4) is obtained by taking the scalar product of the Navier–Stokes equation (1.1) with the velocity field $\boldsymbol{u}$, integrating over space, integrating by parts, applying continuity and the boundary conditions, noting that $\boldsymbol{u}\boldsymbol{\cdot}\partial\boldsymbol{u}/\partial n=0$ and taking the time average in the limit that $T\to\infty$, assuming *a priori* that $\|\boldsymbol{u}\|_2^2$ remains bounded;

(ii) relation (1.5) is obtained by integrating the wall-normal derivative of the laminar velocity profile, $U'(x_2)$, over Ω;

(iii) relation (1.6) is obtained by substituting (1.3) into (1.1), going through a derivation similar to the one leading to (1.4), then applying (1.4) and noting that, by (1.2) and (1.3), it follows that

$$\int_{\Omega} v_1(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} = 0 \quad \forall t; \tag{1.7}$$

(iv) the fact that the underbraced term in relation (1.6) is zero follows from integration by parts, the fact that the second derivative of the laminar velocity profile, U'', is constant, and (1.7).

For details of all four of these derivations, see Bewley & Aamo (2004). Note that (1.6) is related to Onsager's minimum dissipation principle (see Onsager 1931*a*,*b*, 1945). Note also that

(i) the quantity $\left\langle \int_{\Gamma_2^{\pm}} \phi\,(p + \phi^2/2)\, \mathrm{d}\boldsymbol{x} \right\rangle_{\infty}$ may be interpreted as the time-averaged power input applied at the walls, which is the sum of a term related to the kinetic energy input and term related to the pressure $\times$ velocity work done on the flow;

(ii) the quantity $\langle D \rangle_{\infty} U_B V$ may be interpreted as the time-averaged power required to maintain the unsteady controlled flow by the bulk pressure gradient (note that the force applied by the pressure gradient on the bulk fluid is exactly balanced the drag force integrated over the walls in a constant mass flux channel flow);

(iii) the quantity $D_L U_B V$ may be interpreted as the power required to maintain the laminar flow by the bulk pressure gradient.

It follows directly from (1.4), (1.5) and (1.6) that if $\phi \neq 0$ (and thus, by continuity, $\boldsymbol{v} \neq 0$), then

$$\left\langle \int_{\Gamma_2^{\pm}} \phi\,(p + \phi^2/2)\, \mathrm{d}\boldsymbol{x} \right\rangle_{\infty} - [D_L U_B V - \langle D \rangle_{\infty} U_B V] = \left\langle\, \|\nabla \boldsymbol{v}\|_2^2 \,\right\rangle_{\infty} > 0; \tag{1.8}$$

that is the power of the control input applied at the walls is always *larger* than any possible power saved due to drag reduction when $\langle D \rangle_{\infty} < D_L$ for any possible control distribution $\phi(\boldsymbol{x}, t)$.

2. Summary

This paper establishes that, energetically (that is, accounting for the power saved due to drag reduction and the power exerted by application of the control), the 'best' one can do when controlling a turbulent flow in order to maintain that flow with the smallest amount of power is to relaminarize the flow. This result is known as a fundamental performance limitation. The establishment of fundamental performance limitations is a powerful result in the analysis of control systems that often leads to substantial new physical insight. Perhaps the most famous performance limitation in linear systems is known as Bode's integral formula (see Bode 1945) and is often referred to as the 'waterbed' effect: that is if the sensitivity function in a linear feedback control system is reduced at some frequencies, it will necessarily be increased at other frequencies in such a way that the integral of the log of the sensitivity function over all frequencies, known as the Bode integral, is constant. Fundamental performance limitations are notoriously difficult to establish in nonlinear partial differential equation (PDE) systems, as they bound a relevant performance metric of the system over an entire *class* of possible control inputs. The only other fundamental performance limitation currently proven for the Navier–Stokes equation with boundary controls is related to the minimum heat flux in a channel flow and is given in Bewley & Ziane (2007). Note that, in a related work, Fukagata, Kasagi & Sugiyama (2005) presented (but did not derive) an energy balance relation for pipe flow with interior control forcing and homogeneous boundary conditions; that relation is consistent with the present proof.

It is our hope that results of this class will clarify substantially our insight into what can and what cannot be accomplished via control input to fluid systems.

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