A new method for approximating proportional representation in multi-seat elections based on range or approval voting

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This paper develops a new voting system, dubbed Transferable Range Voting (TRV), for approximating proportional representation in multi-seat elections. TRV combines range balloting, in which voters score all candidates independently on an absolute scale, with the key idea of transferable votes, as implemented by the Single Transferable Vote (STV) approach, in which excess voting currency, beyond that necessary to get elected, is transferred to other candidates. The scores that voters assign to candidates in this range balloting approach may be either integers or real numbers over any predefined range $[A, B]$, which TRV rescales to the range $[0, 1]$, and thus includes the important special case of approval balloting, with admissible scores of $\{0, 1\}$ (disapprove/approve) only; the present method is referred to as Transferable Approval Voting (TAV) when applied in this simplified setting. As it is implemented today, STV is based on preferential balloting, in which voters rank order their preferred candidates; this paper thus generalizes the powerful notion of transferable votes, as used by STV, to the range and approval balloting settings for multi-seat elections.

1 Introduction

The selection of voting systems is one of the most consequential foundational underpinnings of any representative-based democratic institution, which must carefully balance majority opinions with minority interests. The very purpose of voting in such institutions is that, when there are different like-minded groups within the electorate who tend to vote for candidates who represent their own dominant interests, then these groups should be represented among the winners of the election, with numbers approximately proportional to their strength within the electorate. This important property is referred to as *proportional representation* [7]. In this regard, it is broadly recognized that many currently-implemented strategies, such as single-seat elections with choose-one balloting methods in highly gerrymandered legislative districts, as implemented in the US, UK, Canada, India, Pakistan, and elsewhere, leave significant room for improvement [6, 1].

For the purpose of this discussion, some definitions help us to get started:

Definition 1. *An* **election** *is a group decision-making process by which a population, or* **electorate***, makes decisions using* **ballots***.* **Electoral reform** *is the process improving the fairness or effectiveness of an election system by modifying the types of ballots selected (see Definition 2), and/or the type of voting system used to tally said ballots (see Definition 3).*

For the purpose of discussion in this paper, the language we will use will specifically indicate the election of **candidates**, in **single-seat** elections, for selecting a single winner, or **multi-seat** elections, for selecting multiple winners. However, elections can also be used for making many other decisions, at the UN, in faculty meetings, for selecting the next books to read in a book club, etc.; the application of election systems to making such other decisions follow in a straightforward manner.

Definition 2. **Ballots** *come in several different types, including:*
• **choose-one** *ballots, which allow voters to select one candidate only;*
• **cumulative** *ballots, in which each voter is assigned* m *voting tokens, which may be distributed by the voter to* m *candidates, or piled up on one or more candidates;*
• **preferential** *ballots, which allow voters to rank order all candidates whom they find acceptable;*
• **approval** *ballots, which allow voters to indicate* $\{0,1\}$ *(disapprove/approve) for each candidate;*
• **combined approval** *ballots, which allow voters to indicate* $\{-1,0,+1\}$ *(against/neutral/for) for each candidate; and*
• **range** *ballots, which allow voters to score each candidate independently on an absolute scale*[1].

The selection of the type of ballot to be used in a given election balances simplicity versus expressiveness of the voters' opinions. Amongst those mentioned in Definition 2, choose-one ballots are the simplest and least expressive, and often engender a mindset akin to an us-versus-them sporting event. Range ballots (e.g., reflecting that candidates $\{a,b,c,d\}$ score, respectively, as $\{10,9,1,0\}$ on a scale of 0 to 10 for a particular voter) can better resolve the voters' opinions than preferential ballots, whereas approval and combined approval ballots (e.g., reflecting that candidates $\{a,b\}$ are acceptable, and candidates $\{c,d\}$ are not, for a particular voter) are somewhat simpler (though, less resolved) than preferential ballots. The use of cumulative ballots is generally not recommended[2].

Once the ballots are cast, a voting system is used to tally them to determine the election outcome.

Definition 3. *A* **voting system** *is an algorithm, or set of tallying rules, used to determine the results of an election using a specific type of ballots (see Definition 2).*

This paper will (in the remainder of §1) survey several notable existing voting systems, each with nuanced differences in what they accomplish. In the remaining sections, we then propose and analyze our new voting system for multi-seat elections using range or approval balloting.

1.1 Single-seat election methods and their properties

As a starting point, consider the setting of single-seat elections. There are several well-defined, mathematically precise properties that are useful in characterizing the many voting methods available for such elections. These properties are studied extensively in the literature on social choice; to set the stage for the discussion that follows, we briefly introduce two of them here (see, e.g., [16]); two additional properties, proportionality and strategyproofness, are discussed further in §1.2 and §2.1:

(A) the *Condorcet* property: any candidate who would win in every two-candidate election against each remaining candidate (in those situations for which this happens to be the case) would also defeat these candidates in the corresponding multi-candidate election.

(B) the *Independent of Irrelevant Alternatives* (IIA) property: assume in an election given the choices $\{a,b\}$, candidate a is preferred over candidate b; introducing a third candidate c and holding a new election over the choices $\{a,b,c\}$, all other factors being the same as before, candidate b can not now be preferred over candidate a.

[1] The scores on range ballots may be real numbers on any predefined interval $[A,B]$ (all such scales are actually equivalent in the end; TRV/TAV shifts the range used on the ballots to $[0,1]$ as its first step) or quantized (e.g., to integers between 1 and 10, or between 0 and 100, or between 0 and 1, or between -1 and +1); approval ballots and combined approval ballots are thus just special cases of range ballots, which together are commonly referred to as *rated* or *cardinal* balloting approaches.

[2] Cumulative voting systems are generally known to be inferior to STV, described in §1.2.1, in terms of the proportionality of the representation ultimately achieved, and are characterized by the important flaw of *strategic voting* mechanisms, in which voters can collude to achieve their desired outcomes disproportionately; see, e.g., [7] for details.

The Condorcet and IIA properties are two precise classifications of voting methods that guarantee specific insensitivity to what is more generally known as the *spoiler effect*. To illustrate, imagine that candidate A is preferred to candidate B in a $\{A, B\}$ matchup, by 50.3% to 49.7%, in a certain election. Now introduce a minor candidate C, with support from only 1.6% of the population in a three-person $\{A, B, C\}$ matchup, and further assume that C draws 70% of his support from voters that would otherwise vote for A in this matchup, 30% of his support from voters that would otherwise vote for B, and all other conditions are the same as before. In this (not-so-hypothetical; see [9]) three-person $\{A, B, C\}$ election:

$49.7 - 0.30 * 1.6 = 49.22\%$ would list B as their first choice.

$50.3 - 0.70 * 1.6 = 49.18\%$ would list A as their first choice, and

1.6% would list C as their first choice.

Thus, if only the first choice of each voter mattered (as in the common *plurality voting system* based on choose-one balloting, which is neither Condorcet nor IIA), B would win, despite the fact that A is actually preferred to B by 0.6% of the electorate, and only 1.6% of the electorate voted for the minor candidate C. If, on the other hand, the voting system used is Condorcet, A would win, as A is preferred to both B and C in individual two-candidate matchups. Similarly, if the voting system used is IIA, then again A would defeat B, because A is preferred to B in an $\{A, B\}$ matchup, so the introduction of C can not change this ordering in an $\{A, B, C\}$ election.

The above tangible example is mentioned here simply to emphasize that the identification of Condorcet and IIA voting systems, which can be something of a subtle exercise, can be significant in the outcome of actual elections aimed at reflecting the will of the electorate, especially in close elections (which are actually quite common in well-functioning democracies); they are not simply an academic exercise. Further, the very willingness of an informed voter to "risk" voting for a "minor" candidate like C, with only a (perceived) outside chance of winning, but who (possibly) much better reflects that voter's own dominant interests in an election, is inherently tied to the voting system implemented. If the voting system is Condorcet and/or IIA, then informed voters can "vote their conscience" (dubbed "sincere" voting), without fear of "wasting" their vote on a candidate with a perceived reduced chance of winning. This assurance has been seen to significantly increase overall voter participation, by up to 12% [3, 8], in actual elections. Without this assurance, overall voter interest and participation in elections is reduced, and the electorate inevitably devolves into a two-party system—and the incomplete representation of the various minority interests of the electorate which a two-party system implies.

With preferential ballots (see Definition 2), each voter rank orders the candidates. There are dozens of techniques available to tally the votes from such ballots in order to determine the winner(s). Some of these methods, such as *Tideman's Ranked Pairs* method [17] and the *Schulze* method [15], are well suited for both the election of a single winner, as well as for developing a rank ordering of the candidates via simple majoritarian (that is, not proportional; see §1.2) considerations. Both the Ranked Pairs method and the Schulze method satisfy the Condorcet property; these two schemes (and the many other available Condorcet schemes for tallying preferential ballots) differ in terms of the procedures by which they reconcile the peculiar but very real possibility that, after all votes are tallied, the electorate prefers A over B, B over C, and C over A, a situation dubbed *Condorcet's voting paradox* [13]. Unfortunately, by *Arrow's impossibility theorem* [2], no preferential voting scheme in which all ballots are accounted for equally can satisfy the IIA property.

With range and approval ballots (again, see Definition 2), voters are asked to score the candidates independently (on an absolute scale), or more simply to mark which candidates they find to be acceptable. When using such ballots, in the single-seat setting, the scores for each candidate are simply summed over all of the ballots, and the candidate with the highest tally wins. Remarkably, this simple approach satisfies both the Condorcet and IIA properties.

1.2 Multi-seat election methods

Small electorates (shareholders, HOAs, etc) generally need to elect a handful of seats to a board in a manner which achieves proportional representation of significant minority interests. In the election of a representative body (congress, parliament, etc.), large electorates in representative democracies generally need to balance such a desire for proportional representation of significant minority interests with geographically local representation. To achieve this balance, now that people no longer travel by horseback, a strong consensus appears to have formed [7, 6, 1, 12] suggesting that approaches based on choose-one ballots (see Definition 2) in small, single-seat districts, the lines of which are gerrymandered [10] by those in power to their own benefit, must be forgone in favor of elections in larger multi-seat geographical districts, with voting schemes that ensure proportional representation within each such multi-seat district. In either case, the development and broad implementation of voting systems capable of achieving the elusive goal of *proportional representation of competing minority interests* in multi-seat elections is of significant interest[3].

In multi-seat elections using range ballots or approval ballots, one might suggest using a simple *majoritarian voting system*, which assigns the candidate with the highest overall sum of scores to the first seat, the candidate with the second-highest sum of scores to the second seat, etc. However, in a strongly polarized election with 60% of the population self-identifying with Party A and 40% of the population self-identifying with Party B, both of which put forth numerous candidates, 100% of the elected candidates would likely be from Party A following a majoritarian approach.

A *proportional representation approach* like that developed in this paper, in contrast, is designed to achieve, in such an example, about 60% of the elected candidates from Party A and 40% from Party B. Theoretical and numerical confirmation that the methods developed in this paper well achieve such proportional representation is reported in §2.1-2.3 [see in particular Lemma 1].

The two major categories of election systems most broadly implemented today to achieve proportional representation in multi-seat elections[4] [1, 12] are *List PR* and the *Single Transferable Vote* (STV). With List PR methods, which are fundamentally tied to political parties, each political party publishes an ordered list of candidates on the ballot. Voters vote for a party, and parties receive seats in proportion to their overall share of votes, with winning candidates taken from these lists, in order of their positions within them. There are many variations of this general approach; in most of them, voters forfeit a significant component of the decision-making process to the political parties responsible for compiling the lists. Alternatively, in the List PR setting, *primary* elections may be held amongst those in the electorate who self-identify with each political party, in order to determine (using, in turn, some mult-seat election method, like the Ranked Pairs or Schulze voting systems mentioned previously) the ordered list of candidates for each political party to be used in the subsequent *general* election. [See also §1.3 for related discussion.]

The Single Transferable Vote (STV), on the other hand, is a popular and well-motivated class of closely-related voting systems [1, 12], leveraging preferential ballots, in which proportional representation in multi-seat elections may be well approximated, without ceding a significant component of the decision making process to political parties. Methods in this class assign one unit of *voting currency* to each ballot, with all of this voting currency initially allocated to each ballot's most preferred candidate; as the tally proceeds and candidates are either elected (by achieving a sufficient

[3] A (perhaps, dubious) counter-argument to this position is given in [5], which suggests that majoritarian (not proportional) elections, and the two-party systems which they engender, empower consumers relative to producers.

[4] A few other voting systems are sometimes used in multi-seat elections, including the *Single Non-Transferable Vote* (SNTV) leveraging choose-one ballots, and systems leveraging cumulative ballots; in both cases, quite simply, the m candidates with the most votes win. Such schemes are generally known to be inferior to STV in terms of the proportionality of the representation ultimately achieved, and are characterized by the important flaw of "strategic voting" mechanisms, in which voters can collude to achieve their desired outcomes disproportionately; see, e.g., [7] for details, and §2.1 for further discussion.

0. Each ballot j is initially assigned $c_j = 1$ voting currency.

1. An "offer" of voting currency is made from each ballot j to the highest-ranked candidate i on that ballot which is still left under consideration; this offer is given simply by the total amount of voting currency which ballot j has left at this iteration.

2a. The offers to each candidate i from all the ballots are then summed, and the highest total offer r compared with a threshold quota required to be named a winner, given by the Droop quota [4] $q = 1+n/(m+1)$; if $r < q$, the candidate with the lowest total offer at this round is removed from further consideration, and the process repeated from step 1.

2b. Otherwise, the candidate with the highest total offer r "cashes in" as the winner of this round. At this point, the amount by which r exceeds q is returned to the respective ballots that elected this candidate in a proportional sense. In other words, if the candidate winning a given round has been offered twice the threshold quota required to be named a winner, the amount of voting currency deducted from each ballot during this round is reduced to half of the amount that it initially offered to the winning candidate.

3. The process then proceeds again from step 1, with the winning candidate at the previous iteration removed from further consideration, until m winning candidates are identified.

Figure 1: The Gregory variant of the STV, in pseudocode.

threshold of voting currency) or eliminated (by not receiving enough voting currency to advance further in the tally), the unused voting currency is appropriately "transferred" to other candidates further down each ballot. The details of how these proportional transfers of votes are performed vary somewhat among different methods in this class; one common variant is summarized in §1.2.1.

In single-seat elections, STV is called *Instant Runoff Voting* (IRV) or *Ranked Choice Voting* (RCV). In such single-seat settings, STV is neither Condorcet nor IIA.

In contrast, recall (from the last paragraph of §1.1) that, in single-seat elections using range or approval ballots, voting systems given simply by summing the scores for each candidate over all of the ballots, and declaring the candidate with the highest score as the winner, is both Condorcet and IIA. This well-conceived approach is the starting point upon which our new TRV/TAV system is based. At each step of the vote tallying process, TRV/TAV (as developed in §2) modifies this approach using the idea of transferable votes, as inspired by STV, to discount at that step the individual ballots that rate highly those candidates that have already won, thereby giving greater consideration, for the steps that remain, to individual ballots that are not yet well represented among the current list of winners. TRV/TAV thus combines the favorable properties of simple voting systems based on range or approval ballots with the proportional representation goals in the multi-seat setting achieved by the use of transferrable votes, as implemented leveraging preferential ballots in STV.

1.2.1 Description of the Gregory variant of the STV algorithm

As introduced above, STV is a simple tallying process for preferential ballots (again, see Definition 2) that is coordinated by the assignment and distribution of fictitious "voting currency". Assuming there are n voters, p candidates, and m seats to be assigned in an election, the Gregory (a.k.a. Senatorial Rules) variant of the STV method [12] is defined as in Figure 1.

It is seen that, in an STV contest, winning candidates need to have both *strong* support, to not be eliminated in step 2a in the early rounds of tallying, and *broad* support, eventually garnering enough voting currency to exceed the threshold q and "cash in", in step 2b. In the $m = 1$ setting of single-seat elections, STV (that is, IRV or RCV) completes as soon as the above algorithm reaches step 2b the first time. In the $m > 1$ setting of multi-seat elections, the goal of proportional representation is well approximated by STV, as those ballots at each later round that are not yet well represented amongst the current winning candidates are counted more heavily in the later rounds of the tallying.

1.3 Primary elections, radicalization vs. pivoting, and polls

Primary + general election approaches, as mentioned in §1.2 in the context of List PR methods for multi-seat elections, are also used extensively in the setting of single-seat elections (as for the US president), where the "list" determined by each party at the primary stage is a single candidate. This approach is far from being either Condorcet or IIA, though, as very different metrics are needed to win the primary and general election contests, especially in the case of single-seat elections when the "independent" voters within the electorate are not necessarily loyal to self-identified party affiliations in the general election, which is common. Typically, candidates vye to "rally their base" in the run-up to the primary election, in order to place well amongst their party's constituents in the primary, then traditionally (at least, to an extent) attempt to "pivot" to more broadly electable postures in the run-up to the general election.

At best, this pivoting on the issues of the day between these two distinct contests is duplicitous. At worst, little to no effort is actually even made to accomplish such a pivot to somewhat more moderate stances after the primary, and the general election (and, the manner of governance thereafter) becomes a polarized "battle" between non-compromising radicalized postures that were solidified by the candidates during the rally-the-base phase of the primary contests.

The deviation of the primary + general election approach from the Condorcet and IIA properties is particularly pronounced when voters are generally lethargic and disenfranchised about voting, often not showing up to vote at all unless a particularly radical candidate, on one end of the political spectrum or the other, is in the running.

We thus now summarize some of the important issues discussed thus far that compel broad populations, not just academics writing papers, to advocate electoral reform:

1. the *spoiler effect*, which is effectively eliminated by Condorcet and IIA methods, as exemplified by the A v. B v. C example highlighted in §1.1,

2. *insincere/strategic voting* [i.e., rather than simply voting for those who you think would do the best job, instead either voting for someone else who you think has a better possibility of winning or, worse, colluding with others to better achieve your desired outcome disproportionately], which is substantially reduced (see §2.1) by implementing tallying algorithms that use transferrable votes, which more robustly achieve proportional representation of minority interests,

3. *voter disenfranchisement* [i.e., reduction in voter turnout] due to the perception that sincere voting would run a high probability of "wasting" one's vote on candidates who might otherwise be viewed as "minor", and the concomitant *inviability of third parties*, both of which are alleviated by the implementation of Condorcet and IIA methods, which assure informed voters that their votes are not wasted when voting sincerely, and

4. a *tendency towards radicalization*, which the primary + general election process appears now to increasingly engender, with the process of "rallying the base" in primary elections tending to solidify uncompromising positions on either end of the political spectrum.

The effect highlighted in issue 4 seems to have intensified in recent years, perhaps as a result of politically-leaning 24-hour news stations repeatedly playing clips of postures made during the primary process, which appear to make it less viable for candidates to pivot from “rallying the base” early in an electoral contest to more centrist personas as the general election (and, the time for actual governance) approaches. The manner in which Condorcet and IIA methods address issues 1, 2, and 3 above highlights their inherent “fairness”, and the manner in which methods that implement transferrable votes (bypassing separate primary and general election contests) address issue 4 highlights what might be identified as their “centrist” tendencies.

It is also important to note that voting systems that effectively address the issues enumerated above do not necessarily benefit those already in political power, and thus electoral reform in favor of Condorcet/IIA single-seat voting systems (e.g., Colorado Prop 131 in 2024), and multi-seat voting systems better achieving proportional representation of minority interests leveraging transferrable votes, are expected to be met with resistance by the leadership of both political parties in existing two-party systems, via any of a number of dubious arguments, notably including the false assertion that anything other than choose-one ballots (see Definition 2) are somehow too “complicated” for voters to clearly express their opinions with (see in particular the discussion related to this point in the last paragraph of §3). To be successful, such a shift would thus likely need to continue to be advocated for by grass-roots voter-based efforts, in districts where direct referendums by voters are viable avenues for change in democratic institutions.

Note also that a focus on *polling*, though giving news stations something less “controversial” or “biased” to dwell upon in the run-up to an election, separate from the issues of the day which frame the important decisions needing to be made, is antithetical to the goal of sincere voting, and instead motivates strategic voting, as discussed at length in §2.1. A shift in the discussion in news outlets, from what decisions or compromises on important issues would best serve the population overall (weighing both majority and minority interests) and how those perspectives align with the platforms of the various candidates, to polls and forecasts regarding which personality will ultimately win a particular contest, and by how much, is thus actually distinctly harmful, as such a focus motivates people to not vote “sincerely” (again, see §2.1). News organizations interested in the survival and flourishing of democratic institutions might do well to reconsider their emphases accordingly. Reforming electoral systems with methods which achieve nearly proportional representation, with reduced effectiveness of strategic voting attempts, also play an important role.

The goal of voting systems, for both single-seat and multi-seat elections, is to represent the “overall will of the electorate”. Existing voting systems achieve this balance well in single-seat elections, using preferential (STV), range, or approval ballots (other voting methods, like those based on choose-one or cumulative ballots, generally do not). This paper proposes a new class of voting systems that addresses the notion of “proportional representation of minority interests” in multi-seat contests (recalling their importance, from the first paragraphs of §1 and §1.2), with reduced effectiveness of insincere voting, by extending the notion of transferrable votes from STV, which uses preferential ballots, to the setting of range and approval ballots, which provide different (more expressive or simpler; see Definition 2) manners in which voters can express their desires.

0. Each ballot j is assigned $c_j = C$ in voting "currency" (nominally, we take $C = 1$, but see §2.1 for discussion of other possibilities).

1. An "offer" of "voting currency" is made from each ballot j to each candidate i, given by the minimum of the amount of voting currency which ballot j has left at this iteration, and the score $s_{i,j}$ which ballot j assigns to candidate i (cf. step 1 of the STV in Figure 1).

2a. The offers to each candidate i from all the ballots are then summed, and the candidate with the highest total offer r "cashes in" and wins this round.

2b. At this point, r is compared with a threshold quota required to be named a TRV winner, initially taken as $q = C \cdot n/m$ (cf. the Droop quota used in step 2 of the STV in Figure 1). The amount by which r exceeds q is then returned to the respective ballots that elected this candidate in a proportional sense. In other words, if the candidate winning a given round has been offered twice the threshold quota required to be named a winner, the amount of voting currency deducted from each ballot during this round is reduced to half of the amount that it initially offered to the winning candidate.

3. The process then proceeds again from step 1, with the winning candidate at the previous iteration removed from further consideration, until m winning candidates are identified.

Figure 2: The TRV/TAV framework in pseudocode, without yet applying q adjustment as discussed in the text and implemented in full in Algorithm 1.

2 Description of the new TRV/TAV algorithm

Assume that there are n voters, p candidates, and m seats to be assigned in a given election. The Transferable Range Voting (TRV) approach proposed here, based on range ballots, tallies votes in a transferable manner, akin to the STV method summarized in §1.2.1, such that proportional representation is ultimately approximated. The score $s_{i,j}$, given to each candidate i by each voter j on their ballot, may initially be an integer or real number over any predefined range. Without loss of generality, we first rescale all votes, by each voter j, to the range $[0, 1]$ in the discussion that follows, and in the corresponding code implementing this algorithm. In the limiting case in which each score used is either a 0 or 1, for disapprove or approve, the method proposed is referred to as Transferable Approval Voting (TAV). Analogous to the description of STV in Figure 1, TRV/TAV is summarized as pseudocode in Figure 2, and in executable Matlab code in Algorithm 1, as a similar process coordinated via transferable voting currency.

Note that, if all n voters ultimately contribute all of their voting currency to winning candidates, the threshold quota required to be named a TRV/TAV winner works out to be precisely $q = C \cdot n/m$, as suggested in step 2b. However, this usually does not work out to be the case, as some ballots only wind up indicating significant support to candidates who ultimately lose. The process described above is thus iterated, with q reduced slightly at each iteration in a simple convergent manner until the "cash in" costs ultimately made by each of the winning candidates work out to be precisely equal. The resulting TRV/TAV method, which incorporates this q adjustment iteration, is illustrated via a simple working Matlab code in Algorithm 1. [In practical tests, this overall q adjustment iteration is usually seen to play a rather minor role in the operation of the TRV/TAV algorithm, as discussed in §2.2.]

Algorithm 1: TRV/TAV, in executable Matlab syntax.

```
function [winners] = TRV(s,m,C,flag)
% function [winners] = TRV(s,m,C,flag)
% INPUTS:  s(i,j) = score that each candidate i is assigned by each voter j (ties ok)
%          m = # of winners in the election
%          C = initial voting currency assigned to each ballot (OPTIONAL, default=1)
%          flag = should each s(:,j) be scaled independently? (OPTIONAL, default=true)
% OUTPUT:  winners = the (ordered) set of m winners in the election
% TESTS:   p=5; n=10; s=rand(p,n); m=3, [winners]=TRV(s,m), pause
%          s=[2 2 2 1 1 1 1 0 0; ...
%             0 1 1 2 2 2 0 0 0; ...
%             0 0 0 0 0 0 2 2 2]; m=2, [winners]=TRV(s,m,1)
% The scores s(i,j) can be defined over any range [A,B] by each voter; the range
% of scores used by each voter is independently (if flag=true) normalized to [0,1]
% during the analysis, so that each voter has the same total impact on the outcome.
% Perhaps take flag=false only if tallying "combined approval" ballots.
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if nargin<4, flag=true; end % specify if scores by each voter be scaled independently
if nargin<3, C=1,       end % initial voting currency assigned to each ballot
[p,n]=size(s)               % p = # of candidates, n = # of voters,
min_s=min(s); max_s=max(s);
if flag % shift scores s(:,j), for each voter j, to be over range [0,1]
   for j=1:n, if max_s(j)==min_s(j), s(1:p,j)=0.5;
              else, s(:,j)=(s(:,j)-min_s(j))/(max_s(j)-min_s(j));
         end, end
else    % shift all scores s to be over range [0,1].  Probably use this only in the
   s=(s-min(min_s))/(max(max_s)-min(min_s));  % case of "combined approval" ballots.
end
if n<11, s, pause, end % print the modified s (if small) to screen, as sanity check
total_cost = C*n;  % initialize total_cost as the total currency
epsilon = 0.00001; % tolerance used to refine threshold quota q
while 1
   q=total_cost/m, % (re-)initialize threshold quota required to be named a winner
   c(1,1:n) = C;   % (re-)initialize voting currency assigned to each ballot
   for k=1:m       % for each winner...
      for i=1:p    % for each candidate...
         offer(i,:)=min(c(1,:),s(i,:));     % amount voters offer to candidate i
         total_offer(i)=sum(offer(i,:));    % total candidate i is offered
      end
      % remove offers to winning candidates, so they don't win again
      for kbar=1:k-1, total_offer(winners(kbar))=0.0; end, total_offer
      [r(k),winners(k)] = max(total_offer);                % identify winners(k)
      cost(k)=min(q,r(k)); % cost(k) is minimum of quota q and winning offer r(k)
      c(1,:)=c(1,:) - offer(winners(k),:)*cost(k)/r(k);    % cash in winners(k)
   end
   previous_total_cost=total_cost; total_cost=sum(cost); % reduce q if necessary
   if abs(total_cost-previous_total_cost) < epsilon*total_cost, break, end
   winners, if n<11, pause, end
end                                                        % repeat until converged
% end function TRV
```

It is noted here that TAV is not the first attempt at proportional representation leveraging approval voting. A few computationally-complex competing algorithms have been suggested, as discussed in [11], based on minisum, minimax, and weighted sum comparisons between ballots. A key reason that TAV is attractive in comparison to these schemes is its relative simplicity.

2.1 Proportionality vs strategyproofness

To illustrate how the TAV and TRV systems work, we now apply them to a few illustrative examples, then summarize what these examples indicate with a couple of definitions and lemmas.

Example 1. Consider an election with $n = 9$ voters[5], $p = 4$ candidates (a, b, c, d), and $m = 3$ winners:

voter 1 approves of candidate a only,
voter 2 approves of candidates $\{a, b\}$,
voters 3-6 approve of candidates $\{a, b, c\}$, and
voters 7-9 approve of candidate d only.

The above approval scores are summarized in matrix form $s(\text{candidate}, \text{voter})$ as follows:

$$s = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{1}$$

Candidates $\{a, b, c, d\}$ are approved by, respectively, $\{6/9, 5/9, 4/9, 3/9\}$ of the voters. By a majoritarian approval argument, candidates $\{a, b, c\}$ should win. However, for "proportional representation of minority interests", candidate d should be one of the 3 winners (because $m/n = 1/3$ of the electorate solely approves of candidate d), along with the most highly approved among $\{a, b, c\}$. Accordingly, TAV with $C = 1$ selects $\{a, b, d\}$. △

Example 2. Consider an election with $n = 9$ voters, $p = 4$ candidates (a, b, c, d), and $m = 3$ winners:

voters 1-6 vote as in Example 1, and
voters 7-9 now approve of candidates $\{c, d\}$.

These approval scores are summarized in matrix form as:

$$s = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{2a}$$

By a majoritarian approval argument, candidates $\{a, b, c\}$ should again win. Since the voters approving the (least-popular) candidate d now also approve the (now, most-popular) candidate c, the argument that the candidate d must be selected, in order to represent their minority interests, is now significantly diminished; accordingly, TAV with $C = 1$ now selects $\{a, b, c\}$.

Note the similarity between (2a) and (1). If, say, voter 7 now misrepresents his sincere objectives described above, and removes his support for candidate c on his ballot (confident, from recent polling, that candidate c is already broadly supported), resulting in scores of

$$s = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \tag{2b}$$

then TAV with $C = 1$ now selects $\{a, c, d\}$, and voter 7 has successfully "manipulated" the election in his favor (recall that he "sincerely" wanted candidates $\{c, d\}$ to win; by *not* approving of candidate c on his ballot, the net result is that *both* of his preferred candidates now get elected). This indicates that TAV in this case is not "strategyproof". △

[5] Or 90 or 9,000 or 9,000,000 voters in roughly the same proportions...

Example 3. Consider an election with $n = 9$ voters, $p = 3$ candidates (a, b, c), and $m = 2$ winners:

voter 1 approves of candidate a only,
voters 2-3 prefer candidate a, with a second choice of candidate b,
voters 4-6 prefer candidate b, with a second choice of candidate a, and
voter 7 prefers candidate c, with a second choice of candidate a, and
voters 8-9 approve of candidate c only.

Depending on the range of available scores available on the ballots, these scores might be summarized in matrix form as something like the following:

$$s = \begin{pmatrix} 1 & 1 & 1 & 0.5 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{3a}$$

By a majoritarian approval argument, candidates $\{a, b\}$ clearly lead. Note that candidate c has nowhere near the threshold of 50% isolated support necessary to be declared the winner in this case by a proportional representation argument. Note also that voter 7 effectively "dilutes" his support of candidate c by also indicating weak support of candidate a. TRV with $C = 1$ selects $\{a, b\}$ in this example.

However, if the mischievous voter 7 again misrepresents his sincere objectives described above, and removes his weak support of candidate a on his ballot, resulting in the scores of

$$s = \begin{pmatrix} 1 & 1 & 1 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \tag{3b}$$

then TRV with $C = 1$ selects[6] $\{a, c\}$. Note that, by removing his (sincere) weak support for candidate a, voter 7 effectively strengthens his support for candidate c as calculated by the TRV algorithm, successfully manipulating the election to achieve his desired outcome, indicating that TRV is also not strategyproof. △

We now formalize the behavior discussed above.

Definition 4. *A* **proportional voting system** *is one that, if there is any group of at least n/m voters who all approve a certain candidate, and do not approve any other candidates, then the candidate that they approve of must be one of the winners, regardless of how the other voters vote.*

Lemma 1. *TRV/TAV taking $C = 1$, as defined by Algorithm 1, is a proportional voting system, as defined in Definition 4.*

Proof. If any block of $\lceil n/m \rceil$ voters approve a certain candidate i, and no other candidates, then by the shift of scores that TRV/TAV performs, to the range $[0, 1]$ (as implemented near the beginning of Algorithm 1), a score of $s_{i,j} = 1$ is given to the approved candidate i by every voter j in this block, and a score of 0 is given to all other candidates by this block. The total offer of voting currency to candidate i by this block of voters is thus $\lceil n/m \rceil$, which (if $C = 1$) meets or exceeds the quota q for cashing in as a winner in the TRV/TAV algorithm at any iteration. □

Note that the result of TAV indicated in Example 1 is consistent with TRV/TAV being a proportional voting system according to Definition 4. This behavior is guaranteed by Lemma 1.

[6]Is this result the "fairest" possible proportional representation of minority interests in this intermediate case? Thoughtful arguments can be made both for and against this answer in this particular case.

The results of Example 2 [contrast the TAV results in the cases of the scores given in (2a) and (2b)] and Example 3 [contrast the TRV results in the cases of the scores given in (3a) and (3b)] indicate that TAV and TRV are both sucsceptible to *strategic voting*. That is, simply by dropping support (even, in the case of TRV, just partial support) of a candidate that a voter sincerely wants elected (trusting the polls, which indicate that the candidate that the voter is dropping support for is well supported by the rest of the electorate, and is thus likely to be elected anyway), the outcome can change in a way that, overall, is preferred by this voter. In such situations, the voter in question is said to have "manipulated" the election by "insincere voting".

Definition 5. *A voting system is said to be susceptible to* **voter manipulation** *by* **insincere voting** *if, simply by dropping support of a candidate that a voter actually wants elected, the outcome of the election can change in a way that, overall, is preferred by this voter. Voting systems that are not susceptible to manipulation in this way are said to said to be* **strategyproof**.

Theorem 1. *No proportional voting system, as defined in Definition 4, is strategyproof, as defined in Definition 5; these two ideal goals are incompatible.*

Proof of Theorem 1, which is quite remarkable, is provided in [14] (see also the references contained therein); this and related results are foundational in the modern literature on social choice. Nonetheless, algorithms that approximate proportional representation of competing minority interests in multiseat elections are still needed.

Note that the very idea of transferrable votes reduces, but does not eliminate, the susceptibility of a given election to voter manipulation, as ballots that express support for a highly popular candidate i are in fact returned substantial voting currency after candidate i cashes in, and thus this returned voting currency may then be offered to other candidates on the ballot by those who voted for candidate i.

Lemma 2. *TRV/TAV taking $C = m$, as defined by Algorithm 1, is simply the majoritarian voting system defined previously, and as such is strategyproof, as defined in Definition 5.*

Proof. In the proportional setting (see Lemma 1), the initial amount of voting currency C assigned to each ballot in TRV/TAV is 1, the maximum score given by any voter to any given candidate once the voting scores are scaled to the range $[0, 1]$. By instead initially taking $C = m$, no ballot will ever run short on voting currency during the TRV/TAV tallying process, at any step of Algorithm 1, as any offer of voting currency is at most $s_{i,j}$ (which is bounded by 1), and only m offers are ever cashed in. Thus, in the $C = m$ case, TRV/TAV reduces to the simple majoritarian voting system defined in §1.2, and any voter dropping support for any candidate amounts simply to decreasing the total support which that candidate receives, but does not change the relative balance of total support given to the other candidates. □

Together with Theorem 1, Lemma 1 and Lemma 2 lead to something of a dilemma. We can (taking $C = 1$ in TRV/TAV) have a proportional voting system, or (taking $C = m$ in TRV/TAV) have a strategyproof voting system, but the TRV/TAV voting system can not at once be both proportional and strategyproof. However, with C, we identify a "knob" that allows us to adjust the behavior of the voting system, from one of these ideals to the other. Selecting C to be somewhere between 1 and m provides a means to *compromise* between the goals of (a) proportional representation, and (b) achieving a strategyproof voting system. As described in §1.2.1, the key idea of the transferrable vote is to elect leaders with both "strong support" (i.e., by majoritarian consideration, achieved by taking C closer to m in TRV/TAV) and "broad support" (i.e., by proportional consideration, achieved by taking C closer to 1 in TRV/TAV). How this balance is best achieved overall is a compromise that needs to be made, and doesn't have a unique answer. With the choice of C, TRV/TAV provides the ability to tune this balance.

2.2 Running TRV/TAV examples

The TRV/TAV approach outlined in §2, with the q adjustment iteration implemented as described (that is, iterating until the "cash in" cost of all winning candidates is approximately equal), is codified in Matlab syntax in Algorithm 1; this code is also available at

https://github.com/tbewley/TRV

Note that Algorithm 1 may be tested quite simply in Matlab or Octave with the examples presented in §2.1 and modifications thereof, as well as with random voting data, as indicated in its leading comments. Tests of this sort show that the iterative refinement of q usually actually changes the value of q only slightly (and, the final result not at all), unless the outcome of the vote, in the multi-seat setting, is very very close (as in the examples presented in §2.1). Interested readers are encouraged to run several tests of this code for the values of $\{p, n, m\}$ of their own particular interest to develop relevant statistics.

2.3 TRV/TAV with votes representative of strong party affiliations

The above Github site also provides novel benchmark test codes for evalutating the behavior of TRV/TAV for populations with strong party affiliations, implementing random votes largely supportive of their political parties of choice, to test the proportionalness of the representation achieved by TRV/TAV when $C = 1$. These test codes, as provided, assume that a given electorate self-identifies as 40% Party A, 30% Party B, 20% Party C, and 10% Party D, and that each of these four political parties has put up 10 candidates to run in an election for a total 10 open seats.

In short, it is found in these examples (which interested readers are encouraged to run for themselves) that, if each voter scores a $w_2 = 0.8$ or more for candidates from his/her own political party, and scores a $w_1 = 0.2$ or less for candidates from other parties, then TRV almost always returns 4 winners that are from Party A, 3 that are from Party B, 2 that are from Party C, and 1 that is from Party D, thus providing proportional representation along party lines. Recall that TRV combines the information from all of the votes on all of the ballots; if w_2 is significantly reduced and/or w_1 is significant increased, reflecting a less polarized electorate in which voters support candidates from other parties in addition to their own, then these distributions begin to change, as well they should.

Modifying the benchmark test code discussed above to instead use approval voting (with all votes restricted to be 0 or 1) is straightforward. Analogous to the results reported above, it is again found that, if each voter scores a 1 for candidates from his/her own political party a sufficiently large (w_2) fraction of the time, and scores a 1 for candidates from a different political party a sufficiently small (w_1) fraction of the time, then, again, TAV nearly always results in representation with proportions which accurately reflect the party affiliations of the electorate itself.

3 Discussion

The Transferable Range Voting (TRV) and Transferable Approval Voting (TAV) methods, coded in Algorithm 1, combine the idea of transferable votes (as implemented in the Gregory variant of STV, using preferential ballots) with the use of range ballots or approval ballots. The aim of these methods is the elusive goal of achieving proportional representation in multi-seat election contests.

The first candidate elected in the TRV/TAV framework (see steps 0, 1, and 2a of the scheme laid out in Figure 2) is chosen via simple majoritarian considerations, and thus this first choice is both Condorcet and IIA (unlike when using STV). Additional winners are sought in a manner which is neither Condorcet nor IIA; instead, additional winners are chosen via a range voting approach which is modified in a manner which discounts individual ballots that rate highly candidates that have already won, thus giving greater consideration to individual ballots that are not yet well represented among the current list of winners. The tests reported herein verify that this idea well achieves the general goal of proportional representation, though the approach is not (indeed, by Theorem 1, can not be) perfectly strategyproof. As described in the last paragraph of §2.1, the amount of voting currency initially assigned to each ballot in the TRV/TAV tally, C, provides an ability to tune the balance between proportional considerations and (strategyproof) majoritarian considerations in the selection of these additional winners.

Note again that the idea of transferrable votes, as implemented in TRV/TAV when $C = 1$, reduces but does not eliminate the susceptibility of a given election to voter manipulation, as ballots that express support for a highly popular candidate i are returned substantial voting currency after candidate i cashes in, and thus this returned voting currency may then be offered to other candidates on the ballot by those who voted for candidate i.

Taking $C = 1$, the threshold quota q required to be named a TRV/TAV winner is close to, but usually slightly less than, n/m, reflecting the fact that most, but not quite all, of the voting currency initially assigned to the individual ballots is "cashed in" during the TRV tallying process. The precise value of q used in TRV is found iteratively; upon convergence, the "costs" associated with "cashing in" by all of the individual winners are essentially equal.

As seen in the benchmark test proposed in §2.3, if the electorate is highly politically polarized (that is, if voters generally score quite high candidates from their own parties, and score quite low candidates from other parties), then the TRV approach with $C = 1$ accurately achieves proportional representation along party lines (subject to inaccuracies due to quantization). If the electorate is less polarized (generally, a desirable situation), the votes of people from different political parties are more tightly integrated during the TRV tallying process, and proportional representation along strict party lines becomes less accurate. In this case, the vote becomes more about the individual candidates themselves, rather than party lines, but the concept of proportional representation when selecting from among these candidates, as distinct from majoritarian representation as achieved by simple range voting, still applies. Analogously, it is seen in the approval case that, if each voter approves of candidates from his/her own political party a sufficiently large fraction of the time, and disapproves of candidates other different political partis a sufficiently small fraction of the time, then TAV with $C = 1$ also results in proportional representation along party lines.

Interestingly, the TRV/TAV method has a built-in quantification of under-represented minority interests. If $n \gg p > m$, which is often the case, the first-choice candidates of all voters can not usually wind up as TRV winners. However, if upon convergence of Algorithm 1 when $C = 1$, the converged threshold quota q works out to be equal or close to $q_0 = n/m$, then most of the voters at least support the pool of winning candidates enough to spend almost all of their voting "currency". The converged value of $R = 1 - q/q_0$ represents the extent to which some voters do not support the pool of candidates that ultimately won the election. A value of R near zero means that almost all voters have identified candidates that they support strongly enough to spend nearly all of their voting currency on, whereas a value of R near, e.g., 0.1 means that, averaged over all of the ballots, 10% of the voting currency associated with each ballot has gone unspent on the pool of winning candidates. A more direct way of quantifying under-represented minority interests is to take various statistics on the unspent voting currency c_i assigned to each ballot i after the voting is complete. With $C = 1$, the mean value $(1/n)\sum_i c_i$ is simply the metric R mentioned above. The max, median, mode, and rms values of c_i, for example, provide alternative measures of the unspent voting currency that might also be of interest. We are unaware of any other voting schemes for proportional representation that have similar metrics for quantifying the under-represented minority interests.

It is important to emphasize that the STV, TRV, and TAV methods are all fairly simple algorithmically, and may be implemented (by modifying the executable Matlab codes included in this article) as a straightforward cellphone app, webpage, Excel spreadsheet, SQL database, etc. The actual balloting mechanisms for voting (rank ordering the candidates in the case of STV, independently scoring the candidates on an (arbitrary) scale of $[A, B]$ in the case of TRV, and independently approving/disapproving the candidates in the case of TAV) are all quite natural to voters. For those actually interested in the inner workings of the tallying algorithms, the notions upon which all three are based (the assignment and transference of "voting currency," an equal amount of which is initially allocated to each ballot) is both transparent and objectively fair.

Finally, we comment that the TAV approach deserves particular attention, as approval ballots (in which voters may mark *all* of their approved candidates with an X) constitute a particularly straightforward extension of choose-one ballots (in which voters may mark only a *single* candidate with an X). Thus, for the important purpose of introducing multi-seat districts (with proportional representation of significant minority interests) to electorates accustomed only to choose-one ballots in heavily gerrymandered single-seat districts and plurality voting systems, as is the long-standing tradition in most of the US, TAV (based on approval ballots) might be easier to introduce than both STV (based on preferential ballots) and TRV (based on range ballots). On the other hand, more sophisticated electorates might well prefer the enhanced degree of voter expression that is afforded, and properly accounted for, using TRV.

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