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A general framework for robust control in fluid mechanics ☆

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Abstract

The application of *optimal* control theory to complex problems in fluid mechanics has proven to be quite effective when complete state information from high-resolution numerical simulations is available [P. Moin, T.R. Bewley, Appl. Mech. Rev., Part 2 47 (6) (1994) S3–S13; T.R. Bewley, P. Moin, R. Temam, J. Fluid Mech. (1999), submitted for publication]. In this approach, an iterative optimization algorithm based on the repeated computation of an adjoint field is used to optimize the controls for finite-horizon nonlinear flow problems [F. Abergel, R. Temam, Theoret. Comput. Fluid Dyn. 1 (1990) 303–325]. In order to extend this infinite-dimensional optimization approach to control externally disturbed flows in which the controls must be determined based on limited noisy flow measurements alone, it is necessary that the controls computed be insensitive to both state disturbances and measurement noise. For this reason, *robust* control theory, a generalization of optimal control theory, has been examined as a technique by which effective control algorithms which are insensitive to a broad class of external disturbances may be developed for a wide variety of infinite-dimensional linear and nonlinear problems in fluid mechanics. An aim of the present paper is to put such algorithms into a rigorous mathematical framework, for it cannot be assumed at the outset that a solution to the infinite-dimensional robust control problem even exists. In this paper, conditions on the initial data, the parameters in the cost functional, and the regularity of the problem are established such that existence and uniqueness of the solution to the robust control problem can be proven. Both linear and nonlinear problems are treated, and the 2D and 3D nonlinear cases are treated separately in order to get the best possible estimates. Several generalizations are discussed and an appropriate numerical method is proposed. ©2000 Elsevier Science B.V. All rights reserved.

Keywords: Robust control; Fluid mechanics; Navier–Stokes

1. Introduction

In its essence, robust control theory [14,18] may be summarized as Murphy's law [9] taken seriously:

If a worst-case system disturbance can *disrupt a controlled closed-loop system, it* will.

When designing a robust controller, therefore, one should *plan* on a finite component of the worst-case disturbance aggravating the system, and design a controller which is suited to handle even this extreme situation. A controller

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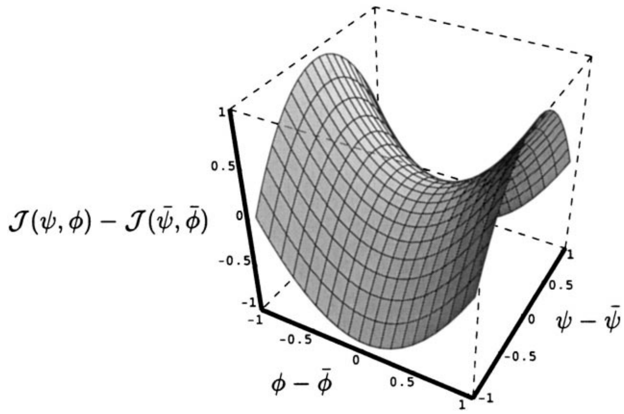

Fig. 1. Schematic of a saddle point representing the neighborhood of a solution to a robust control problem with one scalar disturbance variable ψ and one scalar control variable ϕ. When the robust control problem is solved, the cost function $\mathcal{J}$ is simultaneously maximized with respect to ψ and minimized with respect to ϕ, and a saddle point such as $(\bar\psi,\bar\phi)$ is reached. The present paper formulates an infinite-dimensional extension of this concept, where the cost $\mathcal{J}$ is related to a distributed disturbance ψ and a distributed control ϕ through the solution of the Navier–Stokes equation.

which is designed to work even in the presence of a finite component of the worst-case disturbance will also be robust to a wide class of other possible disturbances which, by definition, are not as detrimental to the control objective as the worst-case disturbance. Thus, the problem of finding a robust control is intimately coupled with the problem of finding the worst-case disturbance in the spirit of a non-cooperative game.

To summarize briefly the robust control approach in the time domain, a cost functional $\mathcal{J}$ describing the control problem at hand is defined that weighs together the (distributed) disturbance ψ, the (distributed) control ϕ, and the flow perturbation $u(\psi,\phi)$ in the domain Ω over the time period of consideration $[0,T]$. The cost functional considered in the present work is of the form

$$\mathcal{J}(\psi,\phi)=\frac{1}{2}\int_0^T\int_\Omega|\mathcal{C}_1u|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_\Omega|\mathcal{C}_2u(x,T)|^2\,\mathrm{d}x \\ -\int_0^T\int_{\partial\Omega}\mathcal{C}_3\nu\frac{\partial u}{\partial n}\cdot\mathbf{r}\,\mathrm{d}\Gamma\,\mathrm{d}t+\frac{1}{2}\int_0^T\int_\Omega[l^2|\phi|^2-\gamma^2|\psi|^2]\,\mathrm{d}x\,\mathrm{d}t. \tag{1.1}$$

This cost functional is simultaneously maximized with respect to the disturbance ψ and minimized with respect to the control ϕ, as illustrated in Fig. 1. The robust control problem is considered to be solved when a saddle point $(\bar\psi,\bar\phi)$ is reached; note that such a solution, if it exists, is not necessarily unique. The dependence of the cost functional $\mathcal{J}$ on the flow perturbation $u=u(\psi,\phi)$ itself is treated in a fairly general form; four cases of particular interest are:

1. $\mathcal{C}_1=d_1\,I$ and $\mathcal{C}_2=\mathcal{C}_3=0\Rightarrow$ regulation of turbulent kinetic energy;
2. $\mathcal{C}_1=d_2\nabla\times$ and $\mathcal{C}_2=\mathcal{C}_3=0\Rightarrow$ regulation of the square of the vorticity;
3. $\mathcal{C}_2=d_3I$ and $\mathcal{C}_1=\mathcal{C}_3=0\Rightarrow$ terminal control of turbulent kinetic energy;
4. $\mathcal{C}_3=d_4I$ and $\mathcal{C}_1=\mathcal{C}_2=0\Rightarrow$ minimization of the time-average skin-friction in the direction $\mathbf{r}$ integrated over the boundary of the domain. Note that $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$ and $\mathbf{r}$ is a given unit vector usually taken as the direction of the mean flow.

All four of these cases, and many others, may be considered in the present framework, and the extension to other cost functionals is straightforward. The dimensional constants d_i (which are the appropriate functions of the kinematic viscosity ν, a characteristic length L_0, and a characteristic velocity U_0), as well as l and γ, are included to make the cost functional dimensionally consistent and to account for the relative weight of each individual term.

It cannot be assumed at the outset that a solution to the infinite-dimensional min/max problem described above even exists. However, it is established in the present paper that for a sufficiently large γ and reasonable requirements on the regularity of the problem (described later in this section), a solution to this min/max problem indeed does exist, with the (finite) magnitudes of the disturbance and the control governed by the scalar parameters γ and l. To accomplish this, we will extend the optimal control setting of Abergel and Temam [1] to analyze the non-cooperative differential game of the robust control setting in which a saddle point $(\bar{\psi}, \bar{\phi})$ is sought. Our approach is based on the results of the existence and characterization of saddle-points in infinite dimensions as given, e.g., in [15].

The optimization of interior forcing profiles (ψ, ϕ) will be examined in detail, first for the linearized Navier–Stokes equation (Section 2), then for the full nonlinear Navier–Stokes equation (Section 3). We will then generalize to the problems of boundary control (Section 4.1), with the possibility of corners in the boundary of the domain Ω, and data assimilation (Section 4.2), in which the initial conditions are optimized to solve an estimation/forecasting problem based on flow measurements on $[0, T]$. Finally, a tractable numerical algorithm for solving all of the robust control problems discussed herein is presented (Section 5).

The numerical approach proposed to solve the robust control problem is based on computations of an $\mathrm{O}(N)$ adjoint field, where N is the number of grid points used to resolve the continuous flow problem. Note that $N \gtrsim \mathrm{O}(10^5)$ for problems of engineering interest today, and this number may be expected only to increase in the future. Computation of the adjoint field is only as difficult as the computation of the flow itself, and thus is a numerically tractable approach to the control problem whenever the computation of the flow itself is numerically tractable. In contrast, control approaches based on the solution of $\mathrm{O}(N^2)$ Riccati equations or Hamilton–Jacobi–Bellman formulations have not been shown to be numerically tractable for discretizations with $N > \mathrm{O}(100)$, and thus are, so far, inadequate to treat many of the problems of interest in fluid mechanics with a sufficient degree of resolution.

1.1. *An intuitive introduction to robust control theory*

Consider the present problem as a differential game between an engineer seeking the “best” control ϕ which stabilizes the flow perturbation with limited control effort and, simultaneously, nature seeking the “maximally malevolent” disturbance ψ which destabilizes the flow perturbation with limited disturbance magnitude [18]. The parameter γ^2 factors into such a competition as a weighting on the magnitude of the disturbance which nature can afford to offer, in a manner analogous to the parameter l^2, which is a weighting on the magnitude of the control which the engineer can afford to offer.

The parameter l^2 may be interpreted as the “price” of the control to the engineer. The $l \to \infty$ limit corresponds to prohibitively “expensive” control, and results in $\phi \to 0$ in the minimization with respect to ϕ for the present problem. Reduced values of l increase the cost functional less upon the application of a control ϕ. A non-zero control results whenever the control ϕ can affect the flow perturbation u in such a way that the net cost functional $\mathcal{J}$ is reduced.

The parameter γ^2 may be interpreted as the “price” of the disturbance to nature. The $\gamma \to \infty$ limit results in $\psi \to 0$ in the maximization with respect to ψ, leading to the optimal control formulation of Abergel and Temam [1] for ϕ alone. Reduced values of γ decrease the cost functional less upon the application of a disturbance ψ. A non-zero disturbance results whenever the disturbance ψ can affect the flow perturbation u in such a way that the net cost functional $\mathcal{J}$ is increased.

Solving for the control ϕ which is effective even in the presence of a disturbance ψ which maximally spoils the control objective is a way of achieving system robustness. A control which works even in the presence of the

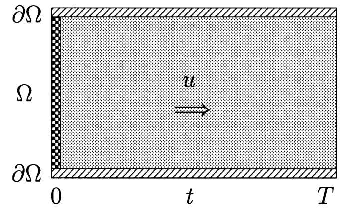

Fig. 2. Schematic representation of the space–time domain over which the flow field u is defined. The arrow indicates the direction in time that the p.d.e. is marched.

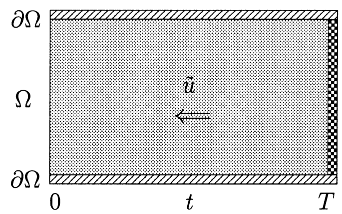

Fig. 3. Schematic representation of the space–time domain over which the adjoint field $\tilde{u}$ is defined. The arrow indicates the direction in time that the p.d.e. is marched.

malevolent disturbance ψ will also be robust to a wide class of other possible disturbances. Put another way, the introduction of the worst-case disturbance in the robust approach is a means of "detuning" the optimal controls. It results in a set of controls which may have somewhat degraded performance when no disturbances are present. However, much greater system robustness (i.e., better performance) is attained in cases for which unknown disturbances are present in the system, and thus the approach is relevant for applications in physical systems, in which unpredictable disturbances are ubiquitous.

In the present systems, for $\gamma < \gamma_0$ for some critical value γ_0 (an upper bound of which is established in this paper), the non-cooperative game is not known to have a finite solution; essentially, the malevolent disturbance wins. The control ϕ corresponding to $\gamma = \gamma_0$ results in a stable system even when nature is on the brink of making the system unstable. However, the control determined with $\gamma = \gamma_0$ is not always the most suitable, as it may result in a very large control magnitude and degraded performance in response to disturbances with structure more benign than the worst-case scenario. In the implementation, variation of l and γ provides the flexibility in the control design which is necessary to achieve the desired trade-offs between Gaussian and worst-case disturbance response and the control magnitude required [8].

1.2. General framework

In Figs. 2 and 3, we identify all possible sources of forcing in the present control problem, which is shown in Sections 2.2 and 3.3 to boil down to a two-point boundary-value problem for a coupled set of p.d.e.s: one for the flow perturbation u and one for an adjoint [1] field $\tilde{u}$. All three possible locations of forcing of the flow problem and all three possible locations of forcing of the adjoint problem are considered in the present framework. By so doing, we establish a general framework in which the robust control approach, discussed herein, can be applied to a wide variety of problems in fluid mechanics.

[1] Note that the adjoint field used in this work represents the sensitivity of the portion of the cost functional $\mathcal{J}$ which depends on u to modification of the forcing (ψ, ϕ) of the flow problem.

The possible regions of forcing in the system defining u are:

1. the right-hand side of the p.d.e., indicated with shading, representing flow control by interior volume forcing, as discussed in Sections 2 and 3 (e.g., externally applied electromagnetic forcing by wall-mounted magnets and electrodes);
2. the boundary conditions, indicated with diagonal stripes, representing flow control by boundary forcing, as discussed in Section 4.1 (e.g., wall transpiration);
3. the initial conditions, indicated with checkerboard, representing optimization of the initial state in a data assimilation framework, as discussed in Section 4.2 (e.g., the weather forecasting problem).

The possible regions of forcing in the system defining $\tilde{u}$, corresponding exactly to the possible domains in which the cost functional $\mathcal{J}$ can depend on u, are:

1. the right-hand side of the p.d.e., indicated with shading, representing regulation of an interior quantity (e.g., turbulent kinetic energy, cases 1 and 2 of Section 1);
2. the boundary conditions, indicated with diagonal stripes, representing regulation of a boundary quantity (e.g., wall skin-friction, case 4 of Section 1);
3. the terminal conditions, indicated with checkerboard, representing terminal control of an interior quantity (e.g., turbulent kinetic energy, case 3 of Section 1).

An interesting singularity arises when considering the terminal control of a boundary quantity such as wall skin-friction. The (inhomogeneous) boundary conditions on the adjoint field for such a case are the same as in the corresponding regulation problem with a delta function applied at time $t = T$.

1.3. Related literature

Robust control of infinite-dimensional linear systems is discussed in a fairly general operator-Riccati setting in [4,16,26]. Though the systems considered in these references are linear (e.g., wave equations) and the issues raised are primarily related to linear operators in infinite dimension, these references provide useful background material for the present discussion; see also [37] for related work in the context of optimal control problems. Most of these references consider the linear case and optimizations over the infinite time horizon, a setting that is effectively analyzed in the frequency domain and referred to as $\mathcal{H}_\infty$ control (with reference to the Hardy spaces on which they are developed and the L^∞-norms of the input–output transfer functions that they bound). The reader is referred to [38] for details from this perspective in the finite-dimensional setting, and the above-mentioned references for details in the infinite-dimensional setting. Of course, there is a wide body of literature concerning generally the theory of control of systems governed by p.d.e.s, including the equations of fluid mechanics: for highlights, the reader is referred for instance to the recent volumes compiled by Banks [2], Banks et al. [3], Gunzburger [19], Lagnese et al. [28], and Sritharan [34].

Robust control of the Navier–Stokes equation in the operator-Riccati ($\mathcal{H}_\infty$) setting is discussed in detail by Barbu and Sritharan [5]. In this work, a robust control problem (on the infinite time horizon) which is γ-suboptimal for the linearized Navier–Stokes equation is stated as the solution of an algebraic Riccati equation, assuming appropriate detectability and stabilizability constraints on the system; then it is shown that this solution is also γ-suboptimal for the full (nonlinear) Navier–Stokes equation in a sufficiently small neighborhood of the origin.

The present analysis differs in several respects. One major difference is that, here, it is not assumed that the system is stabilizable or detectable, a spectral hypothesis difficult to verify in practice. In fact, effective controls may be found by the present non-cooperative optimization approach even if the turbulence may not be subdued entirely in the flow of interest.

As mentioned previously, the robust control problem is solved in the present work by an iterative optimization involving adjoint fields, a numerically tractable approach whenever the computation of the flow itself is numerically tractable (see, e.g., [10], handling 33×10^6 modes in the optimal control framework). In contrast, the largest control Riccati equations solved to date for flow control problems involved Schur decompositions of 280×280 matrices at very high numerical precision (corresponding to a control problem with 140 modes) and relied on the special structure of the problem formulation (stabilization of laminar flow in a plane channel) in order to decouple the control problem at different spatial wave numbers in the two homogeneous directions in the flow [7]; see also an 18×18 Riccati system (corresponding to a control problem with nine modes) in a closely related problem by Joshi et al. [24]. Furthermore, Riccati-based approaches do not extend readily to other geometries (where such decoupling is not present) or to higher-dimensional optimization problems due to their very poor numerical conditioning for large systems. Thus, in the more general setting, iterative adjoint-based control optimizations are preferred over Riccati-based approaches. When optimized over a sufficiently long time horizon T in a receding-horizon predictive control framework, the performance of such schemes (in the optimal control case) has proven to be excellent even in fully turbulent flows (see [10], where the drag of a 3D channel flow is reduced to that of the laminar state from an initial state of fully developed turbulence).

Finally, the present work treats a number of special cases separately (the linear case, the nonlinear 3D case, and the nonlinear 2D case, with interior forcing, boundary forcing, or initial condition optimization) to get sharp estimates on the regularity of the system required in order to be able to prove existence and uniqueness of the solution to an appropriately stated robust control problem.

One of the several applications of the present work is the development of estimator-based feedback control algorithms for flow systems. In order to make such algorithms implementable in hardware in real time, reduced-order models of the flow system which are accurate in the controlled framework (i.e., not just for the uncontrolled system) are a high priority. Much of the pioneering work in the development of reduced-basis representations of nonlinear infinite-dimensional fluid systems is reviewed in the book by Holmes et al. [21]; a review in the context of application to turbulence control is given by Lumley and Blossey [32]. Reduced-basis approaches for related problems are also discussed by Burns and King [11], Cortelezzi and Speyer [12], and Cortelezzi et al. [13]. The latter reference obtains a linear model reduction by truncating those linear eigenmodes with low observability or controllability from the model and report, in their case, a drag reduction to 50% below the laminar level by application of a zero-net mass flux linear controller to a 2D unsteady channel flow. An alternative model reduction strategy was proposed in [6], where it was observed that in the highly non-orthogonal (i.e., nearly defective) situation often encountered in fluid mechanics, model reduction schemes mindful of the transfer function of interest, such as the p, q Markov covariance equivalent realization [36] or optimal Hankel norm approximation [38], are well suited.

1.4. Governing equations

We begin with the Navier–Stokes equation for a flow U in an open domain $\Omega \subset \mathbb{R}^3$ such that, in $\Omega \times (0, \infty)$, we have

$$\frac{\partial U}{\partial t} - \nu \Delta U + (U \cdot \nabla) U + \nabla P = F, \qquad \operatorname{div} U = 0, \qquad U = 0 \quad \text{on } \partial\Omega, \quad U = U_0 \text{ at } t = 0. \tag{1.2}$$

In the bulk of this paper, we focus our attention on the case in which the forcing is applied by way of an interior volume force on the right-hand side of the momentum equation. A stationary or non-stationary solution $U(x, t)$ to this equation with a corresponding forcing $F(x, t)$ will be referred to as the “target” flow for the control problem. (If no target flow is known or given, U and F are taken as zero.)

We are interested in the robust regulation of the deviation of the flow from the desired target (U, F). In Section 2, we consider the control of the linearized equation which models small perturbations (u, f) to the target flow

(U, F) with Dirichlet boundary conditions and known initial conditions such that, in $\Omega \times (0, \infty)$, we have

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)U + (U \cdot \nabla)u + \nabla p = f, \quad \operatorname{div} u = 0, \qquad u = 0 \quad \text{on } \partial\Omega, \quad u = u_0 \quad \text{at } t=0. \tag{1.3}$$

In Section 3, we consider the control of the full nonlinear equation which models large perturbations (u, f) to the target flow (U, F) such that, in $\Omega \times (0, \infty)$, we have

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)U + (U \cdot \nabla)u + (u \cdot \nabla)u + \nabla p = f, \\ &\operatorname{div} u = 0, \quad u = 0 \text{ on } \partial\Omega, \quad u = u_0 \text{ at } t = 0. \end{aligned} \tag{1.4}$$

In Section 4, we will generalize this setting to examine the optimization of boundary controls and the optimization of initial conditions.

1.5. Mathematical setting

Let Ω be a bounded open set of $\mathbb{R}^3$ with boundary $\partial\Omega$, and let $\mathbf{n}$ be the unit outward normal vector to $\partial\Omega$. We denote by $H^s(\Omega)$, for $s \in \mathbb{R}$, the Sobolev spaces constructed on $L^2(\Omega)$, and by $H_0^s(\Omega)$, for $s > 1/2$, the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. Following [35], we set $X = \left\{u \in (C_0^\infty(\Omega))^3; \operatorname{div} u = 0\right\}$, and denote by H (resp. V) the closure of X in $(L^2(\Omega))^3$ (resp. $(H^1(\Omega))^3$); we have

$$H = \{u \in (L^2(\Omega))^3; \operatorname{div} u = 0 \text{ in } \Omega, u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

$$V = \{u \in (H_0^1(\Omega))^3; \operatorname{div} u = 0 \text{ in } \Omega\}.$$

The scalar product on H is denoted by $(u, v) = \int_\Omega u \cdot v \, \mathrm{d}x$, that on V is denoted by $((u, v)) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$, and the associated norms are denoted by $|\cdot|_{L^2(\Omega)}$ and $\|\cdot\|$, respectively. We denote by A the Stokes operator, defined as an isomorphism from V onto the dual V' of V such that, for $u \in V$, Au is defined by

$$\langle Au, v\rangle_{V',V} = ((u, v)), \quad \forall v \in V,$$

where $\langle \cdot, \cdot\rangle_{V',V}$ is the duality bracket between V' and V. The operator A is extended to H as a linear unbounded operator with domain $D(A) = (H^2(\Omega))^3 \cap V$ when $\partial\Omega$ is a C^2 surface; the case of boundary forcing in a domain Ω with corners is treated in Section 4.1. We also recall the Leray–Hopf projector $\mathcal{P}$, which is the orthogonal projector of the non-divergence-free space $(L^2(\Omega))^3$ onto the divergence-free space H. The Stokes operator is defined with this projector such that

$$Au = -\mathcal{P}(\Delta u), \quad \forall u \in D(A). \tag{1.5}$$

We shall denote by $0 < \lambda_1 \le \lambda_2 \le \cdots$ the increasing sequence of the eigenvalues of A. Define the bilinear mapping B by

$$B(u, v) = \mathcal{P}((u \cdot \nabla)v), \quad \forall u, v \in V. \tag{1.6}$$

Note that B is a bilinear mapping from V into V'. Define a continuous trilinear form b on V such that, with $u, v, w \in (H^1(\Omega))^3$, we have

$$b(u, v, w) = \langle B(u, v), w\rangle_{V',V} = \int_\Omega (u \cdot \nabla)v \cdot w \, \mathrm{d}x = \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x,$$

where Einstein's summation is assumed.

1.6. Abstract form of governing equations

The operators A and B may be used to write the Navier–Stokes equation in the "abstract form" useful for mathematical analysis. By application of the Leray projector to (1.3), noting (1.5), (1.6), and that $\mathcal{P}u = u$ and $\mathcal{P}(\nabla p) = 0$, the linearized Navier–Stokes equation may be written in the form

$$\frac{du}{\mathrm{d}t} + \nu Au + B(u, U) + B(U, u) = \mathcal{P}f, \qquad u \in V, \qquad u = u_0 \quad \text{at } t = 0, \tag{1.7}$$

where the regularity required on f, u_0, and U are

$$f \in L^2(0, T; L^2(\Omega)^3), \quad \forall T > 0; \qquad u_0 \in V; \qquad U \in L^\infty(0, T; V) \cap L^2(0, T; D(A)). \tag{1.8}$$

Similarly, application of the Leray projector to the nonlinear form (1.4) gives

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u, U) + B(U, u) + B(u, u) = \mathcal{P}f, \quad u \in V, \quad u = u_0 \text{ at } t = 0. \tag{1.9}$$

1.7. Control framework

In the control framework, the interior forcing f is decomposed into a disturbance $\psi \in L^2(0, T; L^2(\Omega)^3)$ and a control $\phi \in L^2(0, T; L^2(\Omega)^3)$, with $T > 0$, in the spirit of the non-cooperative game discussed in Section 1.1. Thus, we write f as

$$f = B_1\psi + B_2\phi, \tag{1.10}$$

where B_1 and B_2 are taken here as given bounded operators [2] on $(L^2(\Omega))^3$. Only the divergence-free part of the forcing f will affect the evolution of the velocity field u, as seen on the right-hand side of the governing equations (1.7) and (1.9). Thus, in the remainder of this paper, we consider only the divergence-free part of the forcing by writing

$$\mathcal{P}f = \mathcal{P}(B_1\psi + B_2\phi) = \mathcal{B}_1\psi + \mathcal{B}_2\phi, \tag{1.11}$$

where $\mathcal{B}_1 = \mathcal{P}B_1$ and $\mathcal{B}_2 = \mathcal{P}B_2$ are mappings from $(L^2(\Omega))^3$ to H.

The difference $f - \mathcal{P}f$ may be written as the gradient of a scalar and thus will only modify the pressure p in (1.3) and (1.4). As the Navier–Stokes equation in the abstract form is implicitly confined to a divergence-free submanifold of $(L^2(\Omega))^3$, the pressure p may be neglected in the mathematical analysis.

1.8. Important identities and inequalities

We now recall some important properties of the nonlinear operator b, which can be found, for instance, in [27,30,36]. First, we have the orthogonality identity

$$b(u, v, v) = 0, \quad \forall u, v \in V, \tag{1.12}$$

as a consequence of $\operatorname{div} u = 0$, as shown by integration by parts. Moreover, the continuity of the nonlinear mapping in various functional spaces is expressed by the following classical inequalities: there exists a numerical coefficient $C_0 = C_0(\Omega)$ such that

[2] Note that B_1 and B_2 quantify the profile of the forcing inside the domain Ω which results from modification of the disturbance and control variables ψ and ϕ. Generally speaking, ψ and ϕ might be defined on a subdomain of Ω.

$$|b(u,v,w)| \le C_0 \|u\| \|v\|^{1/2} |Av|_{L^2}^{1/2} |w|_{L^2}, \quad \forall u \in V, v \in D(A),\ w \in H,$$

$$|b(u,v,w)| \le C_0 |u|_{L^2}^{1/4} |Au|_{L^2}^{3/4} \|v\| |w|_{L^2}, \quad \forall u \in D(A), v \in V,\ w \in H,$$

$$|b(u,v,w)| \le C_0 |u|_{L^2}^{1/4} \|u\|^{3/4} \|v\| |w|_{L^2}^{1/4} \|w\|^{3/4}, \quad \forall u \in V, v \in V,\ w \in V, \tag{1.13}$$

where C_0 as well as the C_i hereafter denote positive numerical coefficients whose values may be different in each inequality.

Note that the mapping $u \mapsto B(u) = B(u,u)$ is differentiable from V into V'. Its differential is defined by

$$B'(u)v = B(u,v) + B(v,u) = \mathcal{P}\left((u\cdot\nabla)v + (v\cdot\nabla)u\right) \quad \forall v \in V. \tag{1.14}$$

Let $B'(u)^*$ denote the adjoint of $B'(u)$ for the duality between V and V'. (Note that, since V is a Hilbert space and therefore reflexive, the dual of V' can be identified with V.) The adjoint operator $B'(u)^*$ is thus defined by

$$\langle v, B'(u)w\rangle_{V,V'} = \left\langle B'(u)^* v, w\right\rangle_{V',V}. \tag{1.15}$$

It follows from integration by parts [1] that

$$\left\langle B'(u)^* v, w\right\rangle_{V',V} = \int_\Omega \left(\frac{\partial u_i}{\partial x_j} v_i - \frac{\partial v_j}{\partial x_i} u_i\right) w_j \,\mathrm{d}x = \int_\Omega \left((\nabla u)^{\mathrm{T}} \cdot v - (\nabla v)\cdot u\right)\cdot w \,\mathrm{d}x, \tag{1.16}$$

where, again, Einstein's summation is assumed.

The use of adjoint operators to define an appropriate $\mathrm{O}(N)$ adjoint field is central to the development of an efficient numerical algorithm to solve the robust control problem. An iterative approach to the solution of a two-point boundary value problem is presented in Section 5 such that, at each iteration k (on the entire time interval $[0,T]$), a flow field and a corresponding adjoint field are computed to determine the gradients $\mathcal{DJ}/\mathcal{D}\psi$ and $\mathcal{DJ}/\mathcal{D}\phi$ in the vicinity of (ψ^k,ϕ^k). The disturbance ψ^k and the control ϕ^k (again, on the entire time interval $[0,T]$) are then updated, based on this gradient information, and new flow and adjoint fields are computed until the iteration in k converges and a saddle point for the linear or the full nonlinear problem is reached.

The estimates developed in this work in order to prove the existence of a solution to the robust control problem involve integration by parts and the following five fundamental inequalities, which are repeated here for review: the Cauchy–Schwarz inequality $|(u,v)| \le |u|_{L^2}|v|_{L^2}$, Hölder's inequality

$$\int f_1 \cdots f_n \,\mathrm{d}x \le |f_1|_{L^{p_1}} \cdots |f_n|_{L^{p_n}}, \quad |f|_{L^p} = \left(\int |f|^p \,\mathrm{d}x\right)^{1/p}, \quad \frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1,$$

the Poincaré inequalities $|u|_{L^2} \le \lambda_1^{-1/2}\|u\|$ and $\|u\| \le \lambda_1^{-1/2}|Au|_{L^2}$, Young's inequality in the form

$$ab \le \frac{\epsilon}{p}a^p + \frac{\epsilon^{-q/p}}{q}b^q, \quad \forall a,b,\epsilon > 0, \quad \forall p \ \text{s.t.}\ 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and Gronwall's lemma

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le gy + h, \quad \forall t \ge 0 \Rightarrow \ y(t) \le y(0)\exp\left(\int_0^t g(\tau)\,\mathrm{d}\tau\right) + \int_0^t h(s)\exp\left(\int_s^t g(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad \forall t \ge 0.$$

2. Linear problem

As discussed in Section 1, the objective in the robust control problem is to find the best control ϕ in the presence of the disturbance ψ which maximally spoils the control objective. The cost functional considered in the present work, in the mathematical setting described in Section 1.4, is given by

$$\mathcal{J}(\psi,\phi) = \frac{1}{2}\int_0^T |\mathcal{C}_1 u|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}|\mathcal{C}_2 u(T)|^2_{L^2(\Omega)} - \int_0^T \left(\mathcal{C}_3 \nu \frac{\partial u}{\partial n}, \mathbf{r}\right)_{L^2(\partial\Omega)} \mathrm{d}t$$
$$+\frac{1}{2}\int_0^T \left[l^2|\phi|^2_{L^2(\Omega)} - \gamma^2 |\psi|^2_{L^2(\Omega)}\right] \mathrm{d}t, \tag{2.1}$$

where the scalar control parameters γ and l are given, $\mathbf{r}$ is a known vector field on $\partial\Omega$, $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$, and $\mathcal{C}_3^*\mathbf{r}\cdot\mathbf{n}=0$. The operators $\mathcal{C}_1$ and $\mathcal{C}_2$ are unbounded operators on $(L^2(\Omega))^3$ satisfying

$$|\mathcal{C}_i v|^2_{L^2(\Omega)} \le \alpha |v|^2_{L^2(\Omega)} + \beta\|v\|^2 \quad \text{for } i=1,2, \quad \forall v \in V, \tag{2.2}$$

with $\alpha \ge 0$, $\beta \ge 0$, and $\alpha+\beta > 0$, and $\mathcal{C}_3$ is a bounded operator of $(L^2(\partial\Omega))^3$ so that by the trace theorem [31], we have

$$\left|\left(\mathcal{C}_3 \nu \frac{\partial v}{\partial n}, \mathbf{r}\right)_{L^2(\partial\Omega)}\right| \le \kappa\nu\|v\|_{H^{3/2}(\Omega)} \le \kappa'\nu\|v\|^{1/2}|Av|^{1/2}_{L^2(\Omega)} \quad \forall v \in D(A), \tag{2.3}$$

where κ and κ' depend upon $\mathbf{r}$ and Ω. Note that $(\partial/\partial n) : u \mapsto (\partial u/\partial n)|_{\partial\Omega}$ is a mapping from $\mathcal{C}^\infty(\bar{\Omega})$ to $\mathcal{C}^\infty(\partial\Omega)$ (where $\bar{\Omega}$ is the closure of Ω), which extends by continuity to a mapping from $H^s(\Omega)$ to $H^{s-3/2}(\partial\Omega)$ for $s > 3/2$ [31].

In this chapter, the flow u is assumed to be related to the disturbance ψ and the control ϕ through the linearized Navier–Stokes equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + B(u,U) + B(U,u) = \mathcal{B}_1\psi + \mathcal{B}_2\phi, \quad u \in V, \quad u = u_0 \text{ at } t=0, \tag{2.4}$$

which models small deviations of the flow perturbation u from the desired target flow U. The regularity required is given by

$$(\psi,\phi) \in L^2(0,T;L^2(\Omega)^d) \times L^2(0,T;L^2(\Omega)^d); \quad \mathcal{B}_1,\mathcal{B}_2 \in \mathcal{L}(L^2,H); \quad u_0 \in V;$$
$$U \in L^\infty(0,T;V) \cap L^2(0,T;D(A)), \tag{2.5}$$

where the Stokes operator A, the bilinear mapping B, and other notations are described in Section 1.4. The robust control problem to be solved is the following.

Definition 2.1. The disturbance $\bar{\psi} \in L^2(0,T;L^2(\Omega)^d)$ and control $\bar{\phi} \in L^2(0,T;L^2(\Omega)^d)$, and the solution $\bar{u} = u(\bar{\psi},\bar{\phi})$ to (2.4) associated with $(\bar{\psi},\bar{\phi})$ are said to solve the robust control problem when a saddle point $(\bar{\psi},\bar{\phi})$ of the cost functional $\mathcal{J}$ defined in (2.1) is reached such that

$$\mathcal{J}(\psi,\bar{\phi}) \le \mathcal{J}(\bar{\psi},\bar{\phi}) \le \mathcal{J}(\bar{\psi},\phi) \quad \forall(\psi,\phi) \in L^2(0,T;L^2(\Omega)^d) \times L^2(0,T;L^2(\Omega)^d). \tag{2.6}$$

Note that, in this case,

$$\mathcal{J}(\bar{\psi},\bar{\phi}) = \operatorname*{Max}_{\psi\in L^2(0,T;L^2)} \operatorname*{Min}_{\phi\in L^2(0,T;L^2)} \mathcal{J}(\psi,\phi) = \operatorname*{Min}_{\phi\in L^2(0,T;L^2)} \operatorname*{Max}_{\psi\in L^2(0,T;L^2)} \mathcal{J}(\psi,\phi).$$

In this chapter, we will establish both existence and uniqueness of the solution to the robust control problem stated in Definition 2.1, and identify this solution as a function of the unique solution of a two-point boundary value problem for a linear flow/adjoint system on $[0, T]$.

2.1. Existence of a solution to the robust control problem

The proof of the existence of a solution $(\bar{\psi}, \bar{\phi})$ to the robust control problem for the linear case is based on the following existence result.

Proposition 2.2. *Let $\mathcal{J}$ be a functional defined on $X \times Y$, where X and Y are non-empty, closed, unbounded, convex sets. If $\mathcal{J}$ satisfies*
1. $\forall \psi \in X,\ \phi \mapsto \mathcal{J}(\psi, \phi)$ *is convex lower semicontinuous,*
2. $\forall \phi \in Y,\ \psi \mapsto \mathcal{J}(\psi, \phi)$ *is concave upper semicontinuous,*
3. $\exists \psi_0 \in X$ *such that* $\lim_{\|\phi\|_Y \to +\infty} \mathcal{J}(\psi_0, \phi) = +\infty$,
4. $\exists \phi_0 \in Y$ *such that* $\lim_{\|\psi\|_X \to +\infty} \mathcal{J}(\psi, \phi_0) = -\infty$,

then the functional $\mathcal{J}$ has at least one saddle point $(\bar{\psi}, \bar{\phi})$ and

$$\mathcal{J}(\bar{\psi}, \bar{\phi}) = \operatorname*{Min}_{\phi \in Y} \operatorname*{Sup}_{\psi \in X} \mathcal{J}(\psi, \phi) = \operatorname*{Max}_{\psi \in X} \operatorname*{Inf}_{\phi \in Y} \mathcal{J}(\psi, \phi).$$

Proof. The proof is given in [15]. □

We intend to apply Proposition 2.2 to the present problem (2.1) and (2.4) with $X = Y = L^2(0, T; L^2(\Omega)^d)$. In order to establish conditions 1–4 of Proposition 2.2 for the present problem, we need to analyze the evolution equation (2.4).

It can be proven [27,30,36] that given $u_0 \in V$, $U \in L^\infty(0, T; V) \cap L^2(0, T; D(A))$, and $(\psi, \phi) \in L^2(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)^d)$, there exists a unique solution u of (2.4) such that

$$u \in L^\infty(0, T; V) \cap L^2(0, T; D(A)) \quad \forall T > 0.$$

The proof is based on the following a priori estimates. Multiplying (2.4) with u, noting the definition $\mathcal{P} f = \mathcal{B}_1 \psi + \mathcal{B}_2 \phi$ in (1.11) and the orthogonality of (1.12), and applying the Cauchy–Schwarz, Poincaré, and Young's inequalities to the $(\mathcal{P} f, u)$ term and (1.13) to the $b(u, U, u)$ term, we can write

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |u|_{L^2}^2 + \nu \|u\|^2 = (\mathcal{P} f, u) - b(u, U, u) \le \frac{1}{2\nu\lambda_1} |\mathcal{P} f|_{L^2}^2 + \frac{\nu}{2} \|u\|^2 + C_0 \|U\| |u|_{L^2}^{1/2} \|u\|^{3/2},$$

and thus, again noting that C_0 absorbs numerical constants,

$$\frac{\mathrm{d}}{\mathrm{d}t} |u|_{L^2}^2 + \nu \|u\|^2 \le \frac{1}{\nu\lambda_1} |\mathcal{P} f|_{L^2}^2 + C_0 \|U\| |u|_{L^2}^{1/2} \|u\|^{3/2}.$$

An additional application of Young's inequality leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} |u|_{L^2}^2 + \frac{\nu}{2} \|u\|^2 \le \frac{1}{\nu\lambda_1} |\mathcal{P} f|_{L^2}^2 + \frac{C_0}{\nu^3} \|U\|^4 |u|_{L^2}^2. \tag{2.7}$$

Let $M_0(t) = C_0 \nu^{-3} \int_0^t \|U\|^4 \,\mathrm{d}\tau$ and thus $M_0'(t) = C_0 \nu^{-3} \|U(t)\|^4$. Applying Gronwall's lemma to (2.7), we have

$$|u(t)|_{L^2}^2 \le \mathrm{e}^{M_0(t)} |u_0|_{L^2}^2 + \frac{\mathrm{e}^{M_0(t)}}{\nu\lambda_1} \int_0^t |\mathcal{P} f|_{L^2}^2 \,\mathrm{d}s. \tag{2.8}$$

Also, by integrating (2.7) from 0 to t, multiplying by $2/(\nu t)$, and then substituting with the integral of $M_0'(s)$ times (2.8) from 0 to t, we have

$$\frac{1}{t}\int_0^t \|u\|^2\,\mathrm{d}s \le \frac{2}{\nu^2\lambda_1 t}\int_0^t |\mathcal{P}f|_{L^2}^2\,\mathrm{d}s + \frac{2}{\nu t}\int_0^t M_0'(s)|u(s)|_{L^2}^2\,\mathrm{d}s + \frac{2}{\nu t}|u_0|_{L^2}^2$$
$$\le \frac{2\mathrm{e}^{M_0(t)}}{\nu t}|u_0|_{L^2}^2 + \frac{2\mathrm{e}^{M_0(t)}}{\nu^2\lambda_1 t}\int_0^t |\mathcal{P}f|_{L^2}^2\,\mathrm{d}s. \tag{2.9}$$

Similarly, multiplying (2.4) with Au and applying the Cauchy–Schwarz and Young's inequalities to the $(\mathcal{P}f, Au)$ term, we can write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu|Au|_{L^2}^2 = (\mathcal{P}f, Au) - b(u, U, Au) - b(U, u, Au)$$
$$\le \frac{1}{2\nu}|\mathcal{P}f|_{L^2}^2 + \frac{\nu}{2}|Au|_{L^2}^2 + |b(u, U, Au)| + |b(U, u, Au)|,$$

and thus, applying (1.13),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu|Au|_{L^2}^2 \le \frac{1}{\nu}|\mathcal{P}f|_{L^2}^2 + C_0\|U\|^{1/2}|AU|_{L^2}^{1/2}\|u\||Au|_{L^2} + C_0|U|_{L^2}^{1/4}|AU|_{L^2}^{3/4}\|u\||Au|_{L^2}.$$

Additional application of Young's inequality leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\nu}{2}|Au|_{L^2}^2 \le \frac{1}{\nu}|\mathcal{P}f|_{L^2}^2 + \frac{C_0}{\nu}(\|U\||AU|_{L^2} + |U|_{L^2}^{1/2}|AU|_{L^2}^{3/2})\|u\|^2.$$

Let $M_1(t) = C_0\nu^{-1}\int_0^t(\|U\||AU|_{L^2} + |U|_{L^2}^{1/2}|AU|_{L^2}^{3/2})\,\mathrm{d}\tau$. Applying Gronwall's lemma as done in (2.8), we have

$$\|u(t)\|^2 \le \mathrm{e}^{M_1(t)}\|u_0\|^2 + \frac{\mathrm{e}^{M_1(t)}}{\nu}\int_0^t |\mathcal{P}f|_{L^2}^2\,\mathrm{d}s, \tag{2.10}$$

and with a derivation analogous to that leading to (2.9), we have

$$\frac{1}{t}\int_0^t |Au|_{L^2}^2\,\mathrm{d}s \le \frac{2\mathrm{e}^{M_1(t)}}{\nu t}\|u_0\|^2 + \frac{2\mathrm{e}^{M_1(t)}}{\nu^2 t}\int_0^t |\mathcal{P}f|_{L^2}^2\,\mathrm{d}s. \tag{2.11}$$

The a priori estimates (2.8)–(2.11) allow us to characterize the mapping $(\psi, \phi) \mapsto u(\psi, \phi)$. Specifically, we have the following lemma.

Lemma 2.3. *Take $U \in L^\infty(0, T; V) \cap L^2(0, T; D(A))$ and let u be the solution of* (2.4). *The mappings $(\psi, \phi) \mapsto u(\psi, \phi)$ from $L^2(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)^d)$ into $L^2(0, T; V)$ and $(\psi, \phi) \mapsto u(\psi, \phi)|_T$ from $L^2(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)^d)$ into V are affine and continuous. For $u_0 \in V$ and $(\psi, \phi) \in L^2(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)^d)$, the mappings $(\psi, \phi) \mapsto u(\psi, \phi)$ and $(\psi, \phi) \mapsto u(\psi, \phi)|_T$ have Gâteau derivatives $u'(\psi', \phi')$ and $u'(\psi', \phi')|_T$ in every direction $(\psi', \phi') \in L^2(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)^d)$. Finally, the Gâteau derivative $u'(\psi', \phi')$ solves the linear evolution equation*

$$\frac{\mathrm{d}u'}{\mathrm{d}t} + \nu Au' + B'(U)u' = \mathcal{B}_1\psi' + \mathcal{B}_2\phi', \quad u' \in V, \quad u' = 0 \text{ at } t = 0, \tag{2.12}$$

and it follows that $u'(\psi', \phi') \in L^\infty(0, T; V) \cap L^2(0, T; D(A))$.

Proof. The fact that $(\psi,\phi)\mapsto u(\psi,\phi)$ and $(\psi,\phi)\mapsto u(\psi,\phi)|_T$ are affine and continuous follows from the linearity of (2.4) and the a priori estimates (2.8)–(2.11). The existence of the Gâteau derivatives as well as their characterization by (2.12) is proved in [1], to which we refer the reader for more details. □

Remark 2.4. *The solution $u'(\psi',\phi')$ of* (2.12) *can be expressed as a function of ψ' and ϕ' in terms of the Green–Oseen's kernels $G_\psi(x,t,x',t')$ and $G_\phi(x,t,x',t')$* (*see* [27]); *formally, we write*

$$u'(x,t;\psi',\phi')=\int_0^T\int_\Omega\left(G_\psi(x,t,x',t')\psi'(x',t')+G_\phi(x,t,x',t')\phi'(x',t')\right)\,\mathrm{d}x'\,\mathrm{d}t'\stackrel{\triangle}{=}G_\psi\cdot\psi'+G_\phi\cdot\phi'.$$

Notationally, we will denote G_ψ by $\mathcal{D}u/\mathcal{D}\psi$ and G_ϕ by $\mathcal{D}u/\mathcal{D}\phi$, and thus $u'(\psi',\phi')=(\mathcal{D}u/\mathcal{D}\psi)\cdot\psi'+(\mathcal{D}u/\mathcal{D}\phi)\cdot\phi'$. Physically, $u'(\psi',\phi')$ may be thought of as the linear approximation to the perturbation to u when a perturbation ψ' is added to the disturbance ψ and a perturbation ϕ' is added to the control ϕ. A finite dimensional discretization of the Green–Oseen's kernel $G_\psi=\mathcal{D}u/\mathcal{D}\psi$ may be taken as the Jacobian of the discretization of u with respect to the discretization of ψ, as suggested by this notation; an analogous interpretation may be attributed to $G_\phi=\mathcal{D}u/\mathcal{D}\phi$. By causality, $G_\psi(x,t,x',t')=G_\phi(x,t,x',t')=0$ for $t-t'<0$.

With Lemma 2.3 established, we are ready to prove that conditions 1–4 of Proposition 2.2 are satisfied for the present robust control problem.

Lemma 2.5. *Let $u_0\in V$. There exists γ_1 such that, for $\gamma\geq\gamma_1$, we have*
1. $\forall\psi\in L^2(0,T;L^2(\Omega)^d),\ \phi\mapsto\mathcal{J}(\psi,\phi)$ *is convex lower semicontinuous,*
2. $\forall\phi\in L^2(0,T;L^2(\Omega)^d),\ \psi\mapsto\mathcal{J}(\psi,\phi)$ *is concave upper semicontinuous,*
3. $\lim_{|\phi|_{L^2(0,T;L^2(\Omega)^d)}\to+\infty}\mathcal{J}(0,\phi)=+\infty$,
4. $\lim_{|\psi|_{L^2(0,T;L^2(\Omega)^d)}\to+\infty}\mathcal{J}(\psi,0)=-\infty$.

Proof. *Condition* 1. By Lemma 2.3, the map $\phi\mapsto\mathcal{J}(\psi,\phi)$ is lower semicontinuous. As $\phi\mapsto u(\psi,\phi)$ is affine, the convexity of $\phi\mapsto\mathcal{J}(\psi,\phi)$ follows promptly.

Condition 2. By Lemma 2.3, the map $\psi\mapsto\mathcal{J}(\psi,\phi)$ is upper semicontinuous. In order to prove concavity, it is sufficient to show that

$$h(\rho)=\mathcal{J}(\psi+\rho\psi',\phi)$$

is concave w.r.t. ρ, i.e., $h''(\rho)<0$. Taking $u'(\psi',0)=(\mathcal{D}u/\mathcal{D}\psi)\cdot\psi'$, we compute

$$h'(\rho)=\int_0^T\left(\mathcal{C}_1u,\mathcal{C}_1u'\right)_{L^2(\Omega)}\,\mathrm{d}t+\left(\mathcal{C}_2u(T),\mathcal{C}_2u'(T)\right)_{L^2(\Omega)}$$
$$-\int_0^T\left(\mathcal{C}_3\nu\frac{\partial u'}{\partial n},\mathbf{r}\right)_{L^2(\partial\Omega)}\,\mathrm{d}t-\gamma^2\int_0^T\left(\psi+\rho\psi',\psi'\right)_{L^2(\Omega)}\,\mathrm{d}t.$$

It is clear that u' is independent of ρ. Therefore,

$$h''(\rho)=\int_0^T|\mathcal{C}_1u'|^2_{L^2(\Omega)}\,\mathrm{d}t+|\mathcal{C}_2u'(T)|^2_{L^2(\Omega)}-\gamma^2\int_0^T|\psi'|^2_{L^2(\Omega)}\,\mathrm{d}t.$$

Note that $u'=(\mathcal{D}u/\mathcal{D}\psi)\cdot\psi'$ satisfies (2.12) by Lemma 2.3, and thus the a priori estimates (2.8)–(2.11) also follow upon substitution of $u'(\psi',0)$ for $u(\psi,\phi)$, mutatis mutandis. Upon making such a substitution to (2.2) and applying (2.9) and the Poincaré and Cauchy–Schwarz inequalities, noting that $u'=0$ at $t=0$, we find that

$$\int_0^T |\mathcal{C}_1 u'|^2_{L^2(\Omega)}\,\mathrm{d}t \le \alpha \int_0^T |u'|^2_{L^2(\Omega)}\,\mathrm{d}t + \beta \int_0^T \|u'\|^2\,\mathrm{d}t \le \left(\frac{\alpha}{\lambda_1}+\beta\right)\frac{2\mathrm{e}^{M_0(T)}}{\nu^2\lambda_1}\int_0^T |\mathcal{B}_1\psi'|^2_{L^2(\Omega)}\,\mathrm{d}t$$
$$\le \left(\frac{\alpha}{\lambda_1}+\beta\right)\frac{2\mathrm{e}^{M_0(T)}}{\nu^2\lambda_1}|\mathcal{B}_1|^2_{\mathcal{L}(L^2,H)}\int_0^T |\psi'|^2_{L^2(\Omega)}\,\mathrm{d}t,$$

and similarly, by applying (2.2), (2.8) and (2.10) to $u'(\psi',0)$,

$$|\mathcal{C}_2 u'(T)|^2_{L^2(\Omega)} \le \left(\frac{\alpha}{\nu\lambda_1}\mathrm{e}^{M_0(T)}+\frac{\beta}{\nu}\mathrm{e}^{M_1(T)}\right)|\mathcal{B}_1|^2_{\mathcal{L}(L^2,H)}\int_0^T |\psi'|^2_{L^2(\Omega)}\,\mathrm{d}t.$$

Now under the assumption that

$$\gamma^2 \ge \gamma_1^2 = 2\left[\left(\frac{\alpha}{\lambda_1}+\beta\right)\frac{2\mathrm{e}^{M_0(T)}}{\nu^2\lambda_1}+\frac{\alpha}{\nu\lambda_1}\mathrm{e}^{M_0(T)}+\frac{\beta}{\nu}\mathrm{e}^{M_1(T)}\right]|\mathcal{B}_1|^2_{\mathcal{L}(L^2,H)}, \tag{2.13}$$

we have $h''(\rho) < 0$ for $\rho \in \mathbb{R}$. Thus the function h is concave, and the concavity of $\psi \mapsto \mathcal{J}(\psi,\phi)$ follows immediately.

Condition 3. Applying (2.3) to (2.1) and taking $\psi = 0$, we can write

$$\mathcal{J}(0,\phi) \ge \frac{l^2}{2}\int_0^T |\phi|^2_{L^2(\Omega)}\,\mathrm{d}t - \kappa'\nu\int_0^T \|u\|^{1/2}|Au|^{1/2}_{L^2(\Omega)}\,\mathrm{d}t.$$

By the a priori estimates (2.8)–(2.11), it is straightforward to show that there exist constants $C_0 = C_0(T,\Omega,\|u_0\|)$ and $C_1 = C_1(T,\Omega,\|u_0\|)$ such that the latter term is bounded by an expression which is affine in ϕ, i.e.,

$$\kappa'\nu\int_0^T \|u\|^{1/2}|Au|^{1/2}_{L^2}\,\mathrm{d}t \le C_0|\phi|_{L^2(0,T;L^2)} + C_1,$$

and thus

$$\mathcal{J}(0,\phi) \ge \frac{l^2}{2}|\phi|^2_{L^2(0,T;L^2)} - C_0|\phi|_{L^2(0,T;L^2)} - C_1,$$

and condition 3 follows promptly.

Condition 4. Upon substituting (2.9) into (2.2), as done for condition 2, and considering $u(\psi,0)$, it follows that

$$\int_0^T |\mathcal{C}_1 u|^2_{L^2(\Omega)}\,\mathrm{d}t \le \left(\frac{\alpha}{\lambda_1}+\beta\right)\frac{2\mathrm{e}^{M_0(T)}}{\nu^2\lambda_1}|\mathcal{B}_1|^2_{\mathcal{L}(L^2,H)}\int_0^T |\psi|^2_{L^2(\Omega)}\,\mathrm{d}t + C_1,$$

where $C_1 = C_1(T,\Omega,\|u_0\|)$. Similarly, by (2.2), (2.8), and (2.10), we have

$$|\mathcal{C}_2 u(T)|^2_{L^2(\Omega)} \le \left(\frac{\alpha}{\nu\lambda_1}\mathrm{e}^{M_0(T)}+\frac{\beta}{\nu}\mathrm{e}^{M_1(T)}\right)|\mathcal{B}_1|^2_{\mathcal{L}(L^2,H)}\int_0^T |\psi|^2_{L^2(\Omega)}\,\mathrm{d}t + C_1.$$

Finally, we may bound the linear term in $\mathcal{J}(\psi,0)$ by an expression which is affine in ψ with a procedure analogous to that used for condition 3 above. Thus, if $\gamma^2 \ge \gamma_1^2$, we have

$$\mathcal{J}(\psi,0) = \frac{1}{2}\int_0^T |\mathcal{C}_1 u|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}|\mathcal{C}_2 u(T)|^2_{L^2(\Omega)} - \int_0^T \left(\mathcal{C}_3\nu\frac{\partial u}{\partial n},\mathbf{r}\right)_{L^2(\partial\Omega)}\mathrm{d}t - \frac{\gamma^2}{2}\int_0^T |\psi|^2_{L^2(\Omega)}\,\mathrm{d}t$$
$$\le -\frac{\gamma^2}{4}|\psi|^2_{L^2(0,T;L^2)} + C_0|\psi|_{L^2(0,T;L^2)} + C_1,$$

which implies condition 4. □

Putting the statements of this section together, we have established existence of a solution $(\bar{\psi}, \bar{\phi})$ to the robust control problem of Definition 2.1 for the linear case with $\gamma > \gamma_1$.

Theorem 2.6 (Existence of a solution to the robust control problem, linear case). *Assume that $\gamma > \gamma_1$, where*

$$\gamma_1^2 = 2\left[\left(\frac{\alpha}{\lambda_1} + \beta\right)\frac{2\mathrm{e}^{M_0(T)}}{\nu^2\lambda_1} + \frac{\alpha}{\nu\lambda_1}\mathrm{e}^{M_0(T)} + \frac{\beta}{\nu}\mathrm{e}^{M_1(T)}\right]|\mathcal{B}_1|^2_{\mathcal{L}(L^2,H)}.$$

Then there exists a saddle point $(\bar{\psi}, \bar{\phi})$ and $u(\bar{\psi}, \bar{\phi})$ such that

$$\mathcal{J}(\psi, \bar{\phi}) \le \mathcal{J}(\bar{\psi}, \bar{\phi}) \le \mathcal{J}(\bar{\psi}, \phi), \quad \forall(\psi, \phi) \in L^2(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)^d).$$

Proof. The proof follows directly from Lemmas 2.3 and 2.5 and Proposition 2.2. □

It follows from Theorem 2.6 that γ_1 is an upper bound on the critical value γ_0 discussed in Section 1.1.

2.2. Identification of the unique solution to the robust control problem

The existence of a saddle point $(\bar{\psi}, \bar{\phi})$ of the functional $\mathcal{J}$, established in the previous section, implies that

$$\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\psi}(\bar{\psi}, \bar{\phi}) = 0 \quad \text{and} \quad \frac{\mathcal{D}}{\mathcal{D}\phi}(\bar{\psi}, \bar{\phi}) = 0. \tag{2.14}$$

Differentiation of (2.1) leads to expressions for these gradients in weak form:

$$\begin{aligned}\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\psi}(\psi, \phi) \cdot \psi' = {}& \int_0^T \left(\mathcal{C}_1 u, \mathcal{C}_1 \frac{\mathcal{D}u}{\mathcal{D}\psi} \cdot \psi'\right)_{L^2(\Omega)} \mathrm{d}t + \left(\mathcal{C}_2 u(T), \mathcal{C}_2 \frac{\mathcal{D}u(T)}{\mathcal{D}\psi} \cdot \psi'\right)_{L^2(\Omega)} \\ & - \int_0^T \left(\mathcal{C}_3 \nu \frac{\partial}{\partial n}\frac{\mathcal{D}u}{\mathcal{D}\psi} \cdot \psi', \mathbf{r}\right)_{L^2(\partial\Omega)} \mathrm{d}t - \gamma^2 \int_0^T \left(\psi, \psi'\right)_{L^2(\Omega)} \mathrm{d}t,\end{aligned} \tag{2.15}$$

$$\begin{aligned}\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi}(\psi, \phi) \cdot \phi' = {}& \int_0^T \left(\mathcal{C}_1 u, \mathcal{C}_1 \frac{\mathcal{D}u}{\mathcal{D}\phi} \cdot \phi'\right)_{L^2(\Omega)} \mathrm{d}t + \left(\mathcal{C}_2 u(T), \mathcal{C}_2 \frac{\mathcal{D}u(T)}{\mathcal{D}\phi} \cdot \phi'\right)_{L^2(\Omega)} \\ & - \int_0^T \left(\mathcal{C}_3 \nu \frac{\partial}{\partial n}\frac{\mathcal{D}u}{\mathcal{D}\phi} \cdot \phi', \mathbf{r}\right)_{L^2(\partial\Omega)} \mathrm{d}t - l^2 \int_0^T \left(\phi, \phi'\right)_{L^2(\Omega)} \mathrm{d}t.\end{aligned} \tag{2.16}$$

In order to determine the solution to the robust control problem, we define an adjoint state by the equation

$$\begin{aligned}& -\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} + \nu\mathcal{A}^*\tilde{u} + B'(U)^*\tilde{u} = \mathcal{C}_1^*\mathcal{C}_1 u, \\ & \tilde{u}(t) \in V_r = \{v \in (H^1(\Omega))^3; \operatorname{div} v = 0 \text{ in } \Omega, v = \mathcal{C}_3^*\mathbf{r} \text{ on } \partial\Omega\}, \quad t < T, \\ & \tilde{u} = \mathcal{C}_2^*\mathcal{C}_2 u \in H \quad \text{at } t = T,\end{aligned} \tag{2.17}$$

where $\mathcal{A}^*$ is the unbounded operator on $H \cap H^1$ uniquely defined by

$$\left(u', \mathcal{A}^*\tilde{u}\right)_{L^2(\Omega)} = \left(Au', \tilde{u}\right)_{L^2(\Omega)} + \left(\frac{\partial u'}{\partial n}, \tilde{u}\right)_{L^2(\partial\Omega)} \quad \text{for } u' \in D(A),\ \tilde{u} \in H \cap H^1.$$

We have the following lemma.

Lemma 2.7. *Let $u(\psi,\phi)$ be the solution of (2.4) with the regularity on ψ, ϕ, u_0, and U given in* (2.5), *let $u'(\psi',\phi')$ be the solution of* (2.12) *with $(\psi',\phi')\in L^2(0,T;L^2(\Omega)^d)\times L^2(0,T;L^2(\Omega)^d)$, and let $\tilde u$ be the solution of* (2.17). *Then*

$$\int_0^T (\mathcal{C}_1^*\mathcal{C}_1 u, u')_{L^2(\Omega)}\,\mathrm{d}t + (\mathcal{C}_2^*\mathcal{C}_2 u(T), u'(T))_{L^2(\Omega)} - \int_0^T \left(\mathcal{C}_3\nu\frac{\partial u}{\partial n}, \mathbf{r}\right)_{L^2(\partial\Omega)}\mathrm{d}t$$
$$= \int_0^T [(\mathcal{B}_1^*\tilde u, \psi')_{L^2(\Omega)} + (\mathcal{B}_2^*\tilde u, \phi')_{L^2(\Omega)}]\,\mathrm{d}t \tag{2.18}$$

where $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ are the adjoints in $L^2(\Omega)$ of the operators $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proof. The proof follows from integration by parts and the regularity of u, u' and $\tilde u$:

$$\int_0^T (\mathcal{C}_1^*\mathcal{C}_1 u, u')_{L^2(\Omega)}\,\mathrm{d}t + (\mathcal{C}_2^*\mathcal{C}_2 u(T), u'(T))_{L^2(\Omega)} - \int_0^T \left(\mathcal{C}_3\nu\frac{\partial u'}{\partial n}, \mathbf{r}\right)_{L^2(\partial\Omega)}\mathrm{d}t$$
$$= \int_0^T \left(\left[-\frac{\mathrm{d}\tilde u}{\mathrm{d}t} + \nu\mathcal{A}^*\tilde u + B'(U)^*\tilde u\right], u'\right)_{L^2(\Omega)}\mathrm{d}t + (\tilde u(T), u'(T))_{L^2(\Omega)} - \int_0^T \left(\nu\frac{\partial u'}{\partial n}, \tilde u\right)_{L^2(\partial\Omega)}\mathrm{d}t$$
$$= \int_0^T \left(\tilde u, \left[\frac{\mathrm{d}u'}{\mathrm{d}t} + \nu A u' + B'(U)u'\right]\right)_{L^2(\Omega)}\mathrm{d}t = \int_0^T (\tilde u, [\mathcal{B}_1\psi' + \mathcal{B}_2\phi'])_{L^2(\Omega)}\,\mathrm{d}t$$
$$= \int_0^T [(\mathcal{B}_1^*\tilde u, \psi')_{L^2(\Omega)} + (\mathcal{B}_2^*\tilde u, \phi')_{L^2(\Omega)}]\,\mathrm{d}t. \qquad \square$$

Application of (2.18) to (2.15), with $\phi'=0$ and taking $\psi'\in L^2(0,T;L^2(\Omega)^d)$ as arbitrary, leads to an expression for the gradient $\mathcal{DJ}/\mathcal{D}\psi$:

$$\frac{\mathcal{DJ}}{\mathcal{D}\psi}(\psi,\phi) = \mathcal{B}_1^*\tilde u - \gamma^2\psi. \tag{2.19}$$

Similarly, application of (2.18) to (2.16), with $\psi'=0$ and taking $\phi'\in L^2(0,T;L^2(\Omega)^d)$ as arbitrary, leads to an expression for the gradient $\mathcal{DJ}/\mathcal{D}\phi$:

$$\frac{\mathcal{DJ}}{\mathcal{D}\phi}(\psi,\phi) = \mathcal{B}_2^*\tilde u + l^2\phi. \tag{2.20}$$

Now we prove the following important theorem.

Theorem 2.8. *Let $(\bar\psi,\bar\phi)$ be a solution to the robust control problem stated in Definition 2.1. Then*

$$\bar\psi = \frac{1}{\gamma^2}\mathcal{B}_1^*\tilde u \quad\text{and}\quad \bar\phi = -\frac{1}{l^2}\mathcal{B}_2^*\tilde u, \tag{2.21}$$

where $\tilde u$ is found from the solution $(u,\tilde u)$ of the following coupled system:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + B(u,U) + B(U,u) = \left(\frac{1}{\gamma^2}\mathcal{B}_1\mathcal{B}_1^* - \frac{1}{l^2}\mathcal{B}_2\mathcal{B}_2^*\right)\tilde u, \qquad -\frac{\mathrm{d}\tilde u}{\mathrm{d}t} + \nu\mathcal{A}^*\tilde u + B'(U)^*\tilde u = \mathcal{C}_1^*\mathcal{C}_1 u,$$
$$u\in V,\quad \tilde u(t)\in V_r = \{v\in (H^1(\Omega))^3;\ \mathrm{div}\, v = 0 \text{ in } \Omega,\ v = \mathcal{C}_3^*\mathbf{r} \text{ on } \partial\Omega\},$$
$$t<T,\quad u(0)=u_0 \quad\text{and}\quad \tilde u(T) = \mathcal{C}_2^*\mathcal{C}_2 u(T), \tag{2.22}$$

which admits a unique solution for sufficiently large γ. In other words, u and $\tilde u$ are solutions of (2.17) and (2.4) with (ψ,ϕ) replaced by $(\bar\psi,\bar\phi)$.

Proof. The existence of a solution to the robust control problem is established by Theorem 2.6 for $\gamma > \gamma_1$.

A necessary condition for $(\bar{\psi}, \bar{\phi})$ to be a saddle point of the functional $\mathcal{J}$ is given by (2.14). Thus, (2.19) and (2.20) imply that (2.21) follows from the definition of the coupled system developed in this section and summarized in (2.22).

The uniqueness of the solution of the coupled system (2.22) is classical. For $\gamma > \gamma_2(|\mathcal{B}_1|_{\mathcal{L}(L^2,H)}, |\mathcal{B}_2|_{\mathcal{L}(L^2,H)}, l)$, it is clear that $\mathcal{D} = -(\gamma^{-2}\,\mathcal{B}_1\mathcal{B}_1^* - l^{-2}\,\mathcal{B}_2\mathcal{B}_2^*)$ is positive definite. The proof of uniqueness then follows by taking the difference $(u_3, \tilde{u}_3)$ of two solutions $(u_1, \tilde{u}_1)$ and $(u_2, \tilde{u}_2)$, multiplying the u_3 equation by $\tilde{u}_3$ and the $\tilde{u}_3$ equation by u_3, integrating between 0 and T, and then subtracting the two resulting equations. This results in

$$(u_3(T), \tilde{u}_3(T)) - (u_3(0), \tilde{u}_3(0)) + \int_0^T [(\mathcal{D}\tilde{u}_3(t), \tilde{u}_3(t)) + (\mathcal{C}_1^*\mathcal{C}_1 u_3(t), u_3(t))]\,\mathrm{d}t = 0,$$

with $u_3(0) = 0$ and $\tilde{u}_3(T) = \mathcal{C}_2^*\mathcal{C}_2\, u_3(T)$, from which we conclude that $u_3 = \tilde{u}_3 = 0$, and thus the solution is unique. □

To summarize Section 2, for $\gamma > \max(\gamma_1, \gamma_2)$, a solution to the robust control problem stated in Definition 2.1 exists, is unique, and is given by (2.21).

3. Nonlinear problem

In this chapter, we apply the analysis of the previous chapter to the 2D and 3D nonlinear problems written in the form (1.4) or, equivalently, in the abstract form (1.9). We consider the same cost functional as in the previous chapter,

$$\mathcal{J}(\psi, \phi) = \frac{1}{2}\int_0^T |\mathcal{C}_1 u|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}|\mathcal{C}_2 u(T)|^2_{L^2(\Omega)} - \int_0^T \left(\mathcal{C}_3 \nu \frac{\partial u}{\partial n}, \mathbf{r}\right)_{L^2(\partial\Omega)}\,\mathrm{d}t$$
$$+\frac{1}{2}\int_0^T \left[l^2|\phi|^2_{L^2(\Omega)} - \gamma^2|\psi|^2_{L^2(\Omega)}\right]\,\mathrm{d}t. \tag{3.1}$$

Recall that the operators $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ satisfy (2.2), (2.3), and $\mathcal{C}_3^*\mathbf{r}\cdot\mathbf{n} = 0$, and note that $\mathbf{r}$ is as discussed in Section 1. Assume now that u satisfies the nonlinear Navier–Stokes equation (1.9) with (1.11) such that

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + B(u, U) + B(U, u) + B(u, u) = \mathcal{B}_1\psi + \mathcal{B}_2\phi, \quad u \in V, \quad u = u_0 \text{ at } t = 0, \tag{3.2}$$

which models large deviations of the flow perturbation u from the desired target flow U. The regularity required on ψ, ϕ, $\mathcal{B}_1$, $\mathcal{B}_2$, u_0, and U are the same as in (2.5) except with (ψ, ϕ) now confined to non-empty, closed, bounded, convex subsets of $L^2(0, T; L^2(\Omega)^d)$.

The robust control problem to be solved in the nonlinear case is as follows.

Definition 3.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty, closed, bounded, convex subsets of $L^2(0, T; L^2(\Omega)^d)$. The disturbance $\bar{\psi} \in \mathcal{X}$ and control $\bar{\phi} \in \mathcal{Y}$, and the solution $\bar{u} = u(\bar{\psi}, \bar{\phi})$ to (3.2) associated with $(\bar{\psi}, \bar{\phi})$ are said to solve the robust control problem when a saddle point $(\bar{\psi}, \bar{\phi})$ of the cost functional $\mathcal{J}$ defined in (3.1) is reached such that

$$\mathcal{J}(\psi, \bar{\phi}) \le \mathcal{J}(\bar{\psi}, \bar{\phi}) \le \mathcal{J}(\bar{\psi}, \phi) \quad \forall(\psi, \phi) \in \mathcal{X} \times \mathcal{Y}. \tag{3.3}$$

The robust control problem on the bounded domain $(\psi, \phi) \in \mathcal{X} \times \mathcal{Y}$ given in Definition 3.1 is closely related to the robust control problem on the unbounded domain $(\psi, \phi) \in L^2(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2(\Omega)^d)$ given in Definition 2.1.

In this section, we will establish existence of a solution to the robust control problem for the case in which the flow perturbation u is related to the disturbance ψ and the control ϕ through the nonlinear Navier–Stokes equation (3.2). The analysis is similar to that for the linear problem in the previous chapter. Note that Section 3.1 will consider the 3D case, and Section 3.2 will focus specifically on the 2D case, for which stronger results may be established; this is due to the well-known fact that the theory of the Navier–Stokes equation is complete in space dimension 2, which is not the case in space dimension 3.

3.1. Existence of a solution to the robust control problem, 3D case

The proof of existence in this section is similar to that for the linear problem, but is restricted to cases of either "small data" or "small T". The former assumption is valid when attempting to delay transition to turbulence (i.e., keeping a laminar flow laminar) in an externally disturbed flow that is linearly stable but nonlinearly unstable, and is of important engineering significance [7,25]. The latter assumption has been termed the "suboptimal" approximation in earlier work, and has been shown to deliver simple control strategies with reduced long-term performance [6,20,29]. The restrictions used to prove existence of a solution to the robust control problem in the present case are due simply to the current lack of regularity results for the 3D Navier–Stokes equation, not to a shortcoming of the present analysis of the robust control framework.

To proceed under the constraint of a small data condition, we restrict the bounded set $\mathcal{X}\times\mathcal{Y}$ of $L^2(0,T;L^2(\Omega)^d)\times L^2(0,T;L^2(\Omega)^d)$ by the condition that, for any (ψ,ϕ) in $\mathcal{X}\times\mathcal{Y}$, we have

$$|\mathcal{B}_1\psi|^2_{L^2(0,T;H)} + |\mathcal{B}_2\phi|^2_{L^2(0,T;H)} \le C_1\lambda_1^{1/2}\nu^3.$$

We will mention by way of two remarks that this restriction on $\mathcal{X}\times\mathcal{Y}$ may be lifted in the existence proof if T is sufficiently small. The proof of the existence of a solution $(\bar\psi,\bar\phi)$ on $\mathcal{X}\times\mathcal{Y}$ for the nonlinear, 3D case is based on the following existence result.

Proposition 3.2. *Let $\mathcal{J}$ be a functional defined on $\mathcal{X}\times\mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are non-empty, closed, bounded, convex sets. If $\mathcal{J}$ satisfies*
1. $\forall\phi\in\mathcal{Y},\ \psi\mapsto\mathcal{J}(\psi,\phi)$ *is concave upper semicontinuous*,
2. $\forall\psi\in\mathcal{X},\ \phi\mapsto\mathcal{J}(\psi,\phi)$ *is convex lower semicontinuous*,

then the functional $\mathcal{J}$ has at least one saddle point $(\bar\psi,\bar\phi)$ on $\mathcal{X}\times\mathcal{Y}$, which is defined by

$$\mathcal{J}(\bar\psi,\bar\phi) = \operatorname*{Min}_{\phi\in\mathcal{Y}}\ \operatorname*{Max}_{\psi\in\mathcal{X}}\mathcal{J}(\psi,\phi) = \operatorname*{Max}_{\psi\in\mathcal{X}}\ \operatorname*{Min}_{\phi\in\mathcal{Y}}\mathcal{J}(\psi,\phi).$$

Proof. The proof is given in [15]. □

We intend to apply Proposition 3.2 to the present problem (3.1) and (3.2) on the bounded domain $(\psi,\phi)\in\mathcal{X}\times\mathcal{Y}$. In order to establish conditions 1 and 2 of Proposition 3.2 for the present problem, we need to analyze the evolution equation (3.2).

It can be proven (following the framework of Ladyzhenskaya [27], Lions [30] and Temam [35]) that there exists an $R>0$ such that, given $u_0\in V$, $U\in L^\infty(0,T;V)\cap L^2(0,T;D(A))$, and the small data constraints

$$\|u_0\|\le R \quad \text{and} \quad (\psi,\phi)\in\mathcal{X}\times\mathcal{Y}, \tag{3.4}$$

there exists a unique solution u of (3.2) such that

$$u\in L^2(0,T;D(A))\cap L^\infty(0,T;V)\quad \forall T>0.$$

The proof is based on the following a priori estimates. Multiplying (3.2) with u and noting by (1.12) that $b(u,u,u)=0$, the estimates (2.8) and (2.9) follow as in the linear case; specifically,

$$|u(t)|^2_{L^2} \le \mathrm{e}^{M_0(t)}|u_0|^2_{L^2} + \frac{\mathrm{e}^{M_0(t)}}{\nu\lambda_1}\int_0^t |\mathcal{P}f|^2_{L^2}\,\mathrm{d}s \tag{3.5}$$

and

$$\frac{1}{t}\int_0^t \|u\|^2\,\mathrm{d}s \le \frac{2\mathrm{e}^{M_0(t)}}{\nu t}|u_0|^2_{L^2} + \frac{2\mathrm{e}^{M_0(t)}}{\nu^2\lambda_1 t}\int_0^t |\mathcal{P}f|^2_{L^2}\,\mathrm{d}s, \tag{3.6}$$

where $M_0(t) = C_0\nu^{-3}\int_0^t \|U\|^4\,d\tau$.

Also, multiplying (3.2) with Au and following a line of reasoning similar to the linear case, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu|Au|^2_{L^2} \le \frac{1}{\nu}|\mathcal{P}f|^2_{L^2} + C_0(\|U\|^{1/2}|AU|^{1/2}_{L^2} + |U|^{1/4}_{L^2}|AU|^{3/4}_{L^2})\|u\||Au|_{L^2} + C_1\|u\|^{3/2}|Au|^{3/2}_{L^2}.$$

Application of Young's inequality to the first line and the Poincaré inequality to the second line leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\nu}{2}|Au|^2_{L^2} \le \frac{1}{\nu}|\mathcal{P}f|^2_{L^2} + \frac{C_2}{\nu}(\|U\||AU|_{L^2} + |U|^{1/2}_{L^2}|AU|^{3/2}_{L^2})\|u\|^2 + C_3\lambda^{-1/4}\|u\||Au|^2_{L^2}. \tag{3.7}$$

For the remainder of this derivation, we will fix the coefficients C_2 and C_3 in (3.7) not allowing them to further absorb numerical constants.

Now prescribe that the initial conditions u_0, the target flow U, and the forcing $\mathcal{P}f = \mathcal{B}_1\psi + \mathcal{B}_2\phi$ be small, specifically

$$\begin{aligned} &\|u_0\| \le \frac{\nu\lambda_1^{1/4}}{16C_3}, \quad (\|U(t)\||AU(t)|_{L^2} + |U(t)|^{1/2}_{L^2}|AU(t)|^{3/2}_{L^2}) \le \frac{\nu^2\lambda_1}{8C_2}\ \ \forall t,\\ &\int_0^t |\mathcal{P}f|^2_{L^2}\,\mathrm{d}s \le 2\nu\|u_0\|^2\ \ \forall t. \end{aligned} \tag{3.8}$$

Assume t^* is the maximal time such that

$$\|u(t)\| \le 2\|u_0\| \quad \text{for } 0 \le t \le t^*; \tag{3.9}$$

by continuity, this implies that equality is achieved at $t = t^*$ such that

$$\|u(t^*)\|^2 = 4\|u_0\|^2. \tag{3.10}$$

We shall show that this assumption leads to a contradiction, which implies that t^* is instead unbounded, and thus that (3.9) is valid for all t when the conditions of (3.8) are met. Applying conditions (3.9) and (3.8) to (3.7) and using the Poincaré inequality, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\nu}{4}|Au|^2_{L^2} \le \frac{1}{\nu}|\mathcal{P}f|^2_{L^2} \quad \text{for } 0 \le t \le t^*. \tag{3.11}$$

Hence, by the Poincaré inequality and Gronwall's lemma,

$$\|u(t)\|^2 \le \mathrm{e}^{-\nu\lambda_1 t/4}\|u_0\|^2 + \frac{1}{\nu}\int_0^t |\mathcal{P}f|^2_{L^2}\,\mathrm{d}s \quad \text{for } 0 \le t \le t^*,$$

and thus

$$\|u(t)\|^2 \le 3\|u_0\|^2 \quad \text{for } 0 \le t \le t^*.$$

We have arrived at a contradiction with (3.10), and therefore our assumption that t^* is bounded must be false. Thus, given the small data conditions of (3.8), it follows that t^* is unbounded and thus

$$\|u(t)\|^2 \le \mathrm{e}^{-\nu\lambda_1 t/4}\|u_0\|^2 + \frac{1}{\nu}\int_0^t |\mathcal{P}f|_{L^2}^2\,\mathrm{d}s \quad \forall t, \tag{3.12}$$

and from integration of (3.11),

$$\frac{1}{t}\int_0^t |Au|_{L^2}^2\,\mathrm{d}s \le \frac{4}{\nu t}\|u_0\|^2 + \frac{4}{\nu^2 t}\int_0^t |\mathcal{P}f|_{L^2}^2\,\mathrm{d}s \quad \forall t. \tag{3.13}$$

Remark 3.3. *The estimates* (3.12) and (3.13) *also follow without prescribing constraints on* u_0, U, *or* $\mathcal{P}f$ *but instead prescribing small time, specifically*

$$t^* < t_0^* = \left(\frac{1}{\nu}\int_0^T |\mathcal{P}f|_{L^2}^2\,\mathrm{d}s + M'_1 + \frac{C_0}{\nu^3}(1+\|u_0\|^2)^2\right)^{-1}. \tag{3.14}$$

As opposed to the linearized problem studied in Section 2, the mappings $(\psi,\phi)\mapsto u(\psi,\phi)$ and $(\psi,\phi)\mapsto u(\psi,\phi)|_T$ here are not affine. We have only the following.

Lemma 3.4. *Let u be the solution of* (3.2) *with* $u_0 \in V$, $T>0$, $U \in L^\infty(0,T;V)\cap L^2(0,T;D(A))$, *and* (ψ,ϕ) *in the interior of* $\mathcal{X}\times\mathcal{Y}$ *such that the small data conditions of* (3.8) *are satisfied. The mappings* $(\psi,\phi)\mapsto u(\psi,\phi)$ *and* $(\psi,\phi)\mapsto u(\psi,\phi)|_T$ *have Gâteau derivatives* $u'(\psi',\phi')$ *and* $u'(\psi',\phi')|_T$ *in every direction* $(\psi',\phi')\in L^2(0,T;L^2(\Omega)^d)\times L^2(0,T;L^2(\Omega)^d)$. *Further, the Gâteau derivative* $u'(\psi',\phi')$ *solves the linear evolution equation*

$$\frac{\mathrm{d}u'}{\mathrm{d}t} + \nu Au' + B'(U+u)u' = \mathcal{B}_1\psi' + \mathcal{B}_2\phi', \quad u'\in V, \quad u'=0 \text{ at } t=0, \tag{3.15}$$

and it follows that $u'(\psi',\phi')\in L^\infty(0,T;V)\cap L^2(0,T;D(A))$.

Proof. The existence of the Gâteau derivatives as well as their characterization by (2.12) follows as in [1], to which we refer the reader for more details. □

Remark 3.5. *By Remark 3.3, Lemma 3.4 also holds when a condition of small time is satisfied* (*i.e.*, $T<t_0^*$) *in lieu of the small data condition of* (3.8).

Lemma 3.6. *Let $u_0\in V$ satisfying (3.8). There exists* $\gamma_0=\gamma_0(\|u_0\|,\nu,T)$ *such that, for* $\gamma\ge\gamma_0$, *we have*
1. $\forall\phi\in\mathcal{Y}$, $\psi\mapsto\mathcal{J}(\psi,\phi)$ *is strictly concave upper semicontinuous,*
2. $\forall\psi\in\mathcal{X}$, $\phi\mapsto\mathcal{J}(\psi,\phi)$ *is strictly convex lower semicontinuous*.

Proof. First, we note that by Lemma 3.4, and since the norm is lower semicontinuous, the map $\psi\mapsto\mathcal{J}(\psi,\phi)$ is upper semicontinuous, while the map $\phi\mapsto\mathcal{J}(\psi,\phi)$ is lower semicontinuous.

Condition 1. By Lemma 3.4, and since the norm is lower semicontinuous, the map $\psi\mapsto\mathcal{J}(\psi,\phi)$ is upper semicontinuous. In order to prove concavity, it is sufficient to show that

$$h(\rho)=\mathcal{J}(\psi+\rho\psi',\phi)$$

is concave w.r.t. ρ near $\rho=0$, i.e., $h''(0)<0$. Taking $u'(\psi',0)=(\mathcal{D}u/\mathcal{D}\psi)\cdot\psi'$, we compute

$$h'(\rho)=\int_0^T(\mathcal{C}_1u,\mathcal{C}_1u')_{L^2(\Omega)}\,\mathrm{d}t+(\mathcal{C}_2u(T),\mathcal{C}_2u'(T))_{L^2(\Omega)}-\int_0^T\left(\mathcal{C}_3\nu\frac{\partial u'}{\partial n},\mathbf{r}\right)_{L^2(\partial\Omega)}\mathrm{d}t$$
$$-\gamma^2\int_0^T(\psi+\rho\psi',\psi')_{L^2(\Omega)}\,\mathrm{d}t.$$

Note by (3.15) that u' is a solution of

$$\frac{\mathrm{d}u'}{\mathrm{d}t}+\nu Au'+B(U+u,u')+B(u',U+u)=\mathcal{B}_1\psi',\quad u'\in V,\quad u'=0\ \text{at}\ t=0.$$

Noting the similarity to the linear equation (2.4), and following the derivations leading to (2.8)–(2.11), we have the following a priori estimates on u':

$$|u'(t)|_{L^2}^2\le\frac{\mathrm{e}^{M_2(t)}}{\nu\lambda_1}\int_0^t|\mathcal{B}_1\psi'|_{L^2}^2\,\mathrm{d}s,\qquad\int_0^t\|u'\|^2\,\mathrm{d}s\le\frac{2\mathrm{e}^{M_2(t)}}{\nu^2\lambda_1}\int_0^t|\mathcal{B}_1\psi'|_{L^2}^2\,\mathrm{d}s,$$
$$\|u'(t)\|^2\le\frac{\mathrm{e}^{M_3(t)}}{\nu}\int_0^t|\mathcal{B}_1\psi'|_{L^2}^2\,\mathrm{d}s,\qquad\int_0^t|Au'|_{L^2}^2\,\mathrm{d}s\le\frac{2\mathrm{e}^{M_3(t)}}{\nu^2}\int_0^t|\mathcal{B}_1\psi'|_{L^2}^2\,\mathrm{d}s,$$

with

$$M_2(t)=C_0\nu^{-3}\int_0^t\|U+u\|^4\,\mathrm{d}\tau,$$
$$M_3(t)=C_0\nu^{-1}\int_0^t(\|U+u\||A(U+u)|_{L^2}+|U+u|_{L^2}^{1/2}|A(U+u)|_{L^2}^{3/2})\,\mathrm{d}\tau.$$

Similarly, $u''=(\mathcal{D}^2u/\mathcal{D}\psi^2)\cdot\psi'\cdot\hat{\psi}'$ is a solution of

$$\frac{\mathrm{d}u}{\mathrm{d}t}+\nu Au''+B(U+u,u'')+B(u'',U+u)=\mathcal{F},\quad u''\in V,\ \ u''=0\ \text{at}\ t=0,$$

where, taking $u'=(\mathcal{D}u/\mathcal{D}\psi)\cdot\psi'$ and $\hat{u}'=(\mathcal{D}u/\mathcal{D}\psi)\cdot\hat{\psi}'$,

$$\mathcal{F}=-B(\hat{u}',u')-B(u',\hat{u}').$$

The a priori estimates for u'' follow as for u' by replacing $\mathcal{B}_1\psi$ with $\mathcal{F}$.

Taking $\hat{\psi}'=\psi'$, and thus $\hat{u}'=u'$, we now write

$$h''(\rho)=\int_0^T|\mathcal{C}_1u'|_{L^2(\Omega)}^2\,\mathrm{d}t+\int_0^T(\mathcal{C}_1u,\mathcal{C}_1u'')_{L^2(\Omega)}\,\mathrm{d}t+|\mathcal{C}_2u'(T)|_{L^2(\Omega)}^2+(\mathcal{C}_2u(T),\mathcal{C}_2u''(T))_{L^2(\Omega)}$$
$$-\int_0^T\left(\mathcal{C}_3\nu\frac{\partial u''}{\partial n},\mathbf{r}\right)_{L^2(\partial\Omega)}\mathrm{d}t-\gamma^2\int_0^T|\psi'|_{L^2(\Omega)}^2\,\mathrm{d}t.$$

We will show that for sufficiently large γ, the last term dominates in this expression and thus $h''(0)<0$ when $(\psi,\phi)\in\mathcal{X}\times\mathcal{Y}$. First notice by (1.13) that $|\mathcal{F}|_{L^2}\le C_0\|u'\|^{3/2}|Au'|^{1/2}$, and thus that

$$|\mathcal{F}|_{L^2(0,t,L^2(\Omega))}^2\le\frac{C_0\,\mathrm{e}^{[3M_3(t)+M_2(t)]/2}}{\nu^3\lambda_1^{1/2}}(|\mathcal{B}_1|_{\mathcal{L}(L^2,H)}^2)^2(|\psi'|_{L^2(0,t;L^2(\Omega))}^2)^2=M_4(t)(|\psi'|_{L^2(0,t;L^2(\Omega))}^2)^2.$$

Then, given (2.2) and (2.3), and our a priori estimates for u, u', and u'', we have

$$|\mathcal{C}_1u'|_{L^2(0,T;L^2(\Omega))}^2\le\left(\frac{\alpha}{\lambda_1}+\beta\right)\frac{2\mathrm{e}^{M_2(T)}}{\nu^2\lambda_1}|\mathcal{B}_1|_{\mathcal{L}(L^2,H)}^2|\psi'|_{L^2(0,T;L^2(\Omega))}^2=D_1|\psi'|_{L^2(0,T;L^2(\Omega))}^2,$$

$$
\begin{aligned}
|(\mathcal{C}_1 u, \mathcal{C}_1 u'')_{L^2(0,T;L^2(\Omega))}| &\le (|\mathcal{C}_1 u|^2_{L^2(0,T;L^2(\Omega))})^{1/2} (|\mathcal{C}_1 u''|^2_{L^2(0,T;L^2(\Omega))})^{1/2} \\
&\le \left(\frac{\alpha}{\lambda_1} + \beta\right) (3T\|u_0\|^2)^{1/2} \left(\frac{2\mathrm{e}^{M_2(T)}}{\nu^2\lambda_1} M_4(T)\right)^{1/2} |\psi'|^2_{L^2(0,T;L^2(\Omega))} = D_2 |\psi'|^2_{L^2(0,T;L^2(\Omega))},
\end{aligned}
$$

$$
|\mathcal{C}_2 u'(T)|^2_{L^2(\Omega)} \le \left(\frac{\alpha}{\lambda_1} + \beta\right) \frac{\mathrm{e}^{M_3(T)}}{\nu} |\mathcal{B}_1|^2_{\mathcal{L}(L^2,H)} |\psi'|^2_{L^2(0,T;L^2(\Omega))} = D_3 |\psi'|^2_{L^2(0,T;L^2(\Omega))},
$$

$$
\begin{aligned}
|(\mathcal{C}_2 u(T), \mathcal{C}_2 u''(T))_{L^2(\Omega)}| &\le (|\mathcal{C}_2 u|^2_{L^2(\Omega)})^{1/2} (|\mathcal{C}_2 u''|^2_{L^2(\Omega)})^{1/2} \\
&\le \left(\frac{\alpha}{\lambda_1} + \beta\right) (3\|u_0\|^2)^{1/2} \left(\frac{\mathrm{e}^{M_3(T)}}{\nu} M_4(T)\right)^{1/2} |\psi'|^2_{L^2(0,T;L^2(\Omega))} = D_4 |\psi'|^2_{L^2(0,T;L^2(\Omega))},
\end{aligned}
$$

$$
\begin{aligned}
\left|\left(\mathcal{C}_3 \nu \frac{\partial u''}{\partial n}, \mathbf{r}\right)_{L^2(0,T;L^2(\partial\Omega))}\right| &\le \kappa' \nu T^{1/2} |u''|^{1/2}_{L^2(0,T;V)} |u''|^{1/2}_{L^2(0,T;D(A))} \\
&\le \frac{\kappa'(2T)^{1/2}\,\mathrm{e}^{[M_2(T)+M_3(T)]/4}}{\lambda_1^{1/4}} M_4^{1/2}(T) |\psi'|^2_{L^2(0,T;L^2(\Omega))} = D_5 |\psi'|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

Thus, under the assumption that

$$
\gamma^2 > \gamma_3^2 = D_1 + D_2 + D_3 + D_4 + D_5, \tag{3.16}
$$

we have

$$
h''(0) \le (\gamma_3^2 - \gamma^2) |\psi'|^2_{L^2(0,T;L^2(\Omega))} < 0 \qquad \forall\, \psi' \neq 0. \tag{3.17}
$$

The strict concavity of $\psi \mapsto \mathcal{J}(\psi, \phi)$ follows immediately.

Condition 2. By Lemma 3.4, and since the norm is lower semicontinuous, the map $\phi \mapsto \mathcal{J}(\psi, \phi)$ is upper semicontinuous. In order to prove convexity, it is sufficient to show that

$$
g(\rho) = \mathcal{J}(\psi, \phi + \rho\phi')
$$

is convex w.r.t. ρ near $\rho = 0$, i.e., $g''(0) > 0$. Note that

$$
\begin{aligned}
g''(\rho) = &\int_0^T |\mathcal{C}_1 u'|^2_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T (\mathcal{C}_1 u, \mathcal{C}_1 u'')_{L^2(\Omega)}\,\mathrm{d}t + |\mathcal{C}_2 u'(T)|^2_{L^2(\Omega)} + (\mathcal{C}_2 u(T), \mathcal{C}_2 u''(T))_{L^2(\Omega)} \\
&- \int_0^T \left(\mathcal{C}_3 \nu \frac{\partial u''}{\partial n}, \mathbf{r}\right)_{L^2(\partial\Omega)} \mathrm{d}t + l^2 \int_0^T |\phi'|^2_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned}
$$

Also note that the a priori estimates on $u' = (\mathcal{D}u/\mathcal{D}\phi) \cdot \phi'$ and $u'' = (\mathcal{D}^2 u/\mathcal{D}\phi^2) \cdot \phi' \cdot \hat{\phi}'$ and the bounds on the various terms of $g''(\rho)$ follow immediately as in the proof of condition 1 with $\mathcal{B}_2\phi'$ replacing $\mathcal{B}_1\psi'$ everywhere. Thus, under the assumption that

$$
l^2 > l_1^2 = D_2 + D_4 + D_5, \tag{3.18}
$$

we have

$$
g''(0) \ge (l^2 - l_1^2) |\phi'|^2_{L^2(0,T;L^2(\Omega))} > 0 \quad \forall \phi' \neq 0. \tag{3.19}
$$

The strict convexity of $\phi \mapsto \mathcal{J}(\psi, \phi)$ follows immediately. □

Putting the statements of this section together, we have established existence of a solution $(\bar{\psi}, \bar{\phi})$ to the robust control problem of Definition 3.1 for the 3D nonlinear case with sufficiently large γ and l and sufficiently small data or sufficiently small T.

Theorem 3.7 (Existence of a solution to the robust control problem, nonlinear 3D case). *Assume that $\gamma > \gamma_3$ and $l > l_1$, where γ_3 is defined as in* (3.16) *and l_1 is defined as in* (3.18), *and that either the small T constraint is satisfied, $T < t_0^*$, or that u_0, U, and $\mathcal{P}f$ are sufficiently small such that the small data constraints in* (3.8) *are satisfied. Then there exists a saddle point $(\bar{\psi}, \bar{\phi})$ on $\mathcal{X} \times \mathcal{Y}$ and an associated $\bar{u} = u(\bar{\psi}, \bar{\phi})$ such that*

$$\mathcal{J}(\psi, \bar{\phi}) \le \mathcal{J}(\bar{\psi}, \bar{\phi}) \le \mathcal{J}(\bar{\psi}, \phi) \quad \forall (\psi, \phi) \in \mathcal{X} \times \mathcal{Y}.$$

Proof. The proof follows promptly from Lemmas 3.4 and 3.6 and Proposition 3.2. □

3.2. Existence of a solution to the robust control problem, 2D case

The proof of the existence of a robust control solution $(\bar{\psi}, \bar{\phi})$ for the nonlinear problem in the 2D case is similar to that for the 3D case in the previous section with no small data or small T restriction. The improvement is due to the existence of improved versions of the inequalities (1.13) in the 2D case; specifically, we have

$$|b(u, v, w)| \le C_0 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|_{L^2}^{1/2} |w|_{L^2}, \quad \forall u \in V, \ \ v \in D(A), \ \ w \in H,$$

$$|b(u, v, w)| \le C_0 |u|_{L^2}^{1/2} |Au|_{L^2}^{1/2} \|v\| |w|_{L^2}, \quad \forall u \in D(A), \ \ v \in V, \ \ w \in H,$$

$$|b(u, v, w)| \le C_0 |u|_{L^2}^{1/2} \|u\|^{1/2} \|v\| |w|_{L^2}^{1/2} \|w\|^{1/2}, \quad \forall u \in V, \ \ v \in V, \ \ w \in V.$$

We have, as in the 3D case (without any small data restriction),

$$|u(t)|_{L^2}^2 \le \mathrm{e}^{M_0(t)} |u_0|_{L^2}^2 + \frac{\mathrm{e}^{M_0(t)}}{\nu \lambda_1} \int_0^t |\mathcal{P}f|_{L^2}^2 \, \mathrm{d}s,$$

$$\frac{1}{t} \int_0^t \|u\|^2 \, \mathrm{d}s \le \frac{2\mathrm{e}^{M_0(t)}}{\nu t} |u_0|_{L^2}^2 + \frac{2\mathrm{e}^{M_0(t)}}{\nu^2 \lambda_1 t} \int_0^t |\mathcal{P}f|_{L^2}^2 \, \mathrm{d}s,$$

where $M_0(t) = C_0 \nu^{-3} \int_0^t \|U\|^4 \, \mathrm{d}\tau$. Multiplying (3.2) with Au, we now have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2 + \nu |Au|_{L^2}^2 \le \frac{1}{\nu} |\mathcal{P}f|_{L^2}^2 + C_0 (|U|^{1/2} |AU|_{L^2}^{1/2} + |U|_{L^2}^{1/2} |AU|_{L^2}^{1/2}) \|u\| |Au|_{L^2} + C_1 |u|^{1/2} \|u\| |Au|_{L^2}^{3/2}.$$

Applying Young's inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2 + \nu |Au|_{L^2}^2 \le \frac{1}{\nu} |\mathcal{P}f|_{L^2}^2 + \frac{C_0}{\nu} (|U| |AU|_{L^2} + |U|_{L^2} |AU|_{L^2}) \|u\|^2 + \frac{C_1}{\nu^3} |u|^2 \|u\|^4.$$

Set $g(t) = (C_0/\nu)(|U| |AU|_{L^2} + |U|_{L^2} |AU|_{L^2}) + (C_1/\nu^3) |u|^2 \|u\|^2$ and note that

$$\int_0^t g(\tau) \, \mathrm{d}\tau \le M_6(t) + \sup_{0 \le \tau < t} |u(\tau)|_{L^2}^2 \int_0^t \|u\|^2 \, \mathrm{d}s \le M_6(t) + \frac{2}{\nu} \left(\mathrm{e}^{M_0(t)} |u_0|_{L^2}^2 + \frac{\mathrm{e}^{M_0(t)}}{\nu \lambda_1} \int_0^t |\mathcal{P}f|_{L^2}^2 \, \mathrm{d}s \right)^2 = M_5(t),$$

where $M_6(t) = C_0/\nu\int_0^t (|U||AU|_{L^2} + |U|_{L^2}|AU|_{L^2})\,\mathrm{d}s$. It follows by Gronwall's lemma that

$$\|u(t)\|^2 \le \mathrm{e}^{M_5(t)}\|u_0\|^2 + \frac{\mathrm{e}^{M_5(t)}}{\nu}\int_0^t |\mathcal{P}f|_{L^2}^2\,\mathrm{d}s.$$

Using this estimate of $\|u(t)\|^2$, which has been established with no small data or small T constraint, sufficient conditions on γ and l for the existence of a solution to the robust control problem follow immediately as in the 3D case.

Theorem 3.8 (Existence of a solution to the robust control problem, nonlinear 2D case). *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are non-empty, closed, bounded, convex subsets of $L^2(0,T;L^2(\Omega)^d)$ and that $\gamma > \gamma_4$ and $l > l_2$, where $\gamma_4 = \gamma_4(\mathcal{X},\mathcal{Y})$ and $l_2 = l_2(\mathcal{X},\mathcal{Y})$ are defined with a procedure identical to that in the 3D case but with the modified estimates given above. Then there exists a saddle point $(\bar\psi,\bar\phi)$ on $\mathcal{X}\times\mathcal{Y}$ and an associated $\bar u = u(\bar\psi,\bar\phi)$ such that*

$$\mathcal{J}(\psi,\bar\phi) \le \mathcal{J}(\bar\psi,\bar\phi) \le \mathcal{J}(\bar\psi,\phi) \quad \forall(\psi,\phi)\in\mathcal{X}\times\mathcal{Y}.$$

Proof. Follows promptly as in the 3D case with the modified estimates given above. □

3.3. Identification of the gradients to determine the unique solution

Now we prove the main result of this chapter.

Theorem 3.9. *For sufficiently large γ and l, the solution to the robust control problem stated in Definition 3.1 exists and is unique. Further, the gradients of the cost functional $\mathcal{J}(\psi,\phi)$ in* (3.1) *for any $(\psi,\phi)\in\mathcal{X}\times\mathcal{Y}$ are given by*

$$\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\psi}(\psi,\phi) = \mathcal{B}_1^*\tilde u - \gamma^2\psi \quad \textit{and} \quad \frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi}(\psi,\phi) = \mathcal{B}_2^*\tilde u + l^2\phi, \tag{3.20}$$

where $\tilde u$ is found from the solution $(u,\tilde u)$ of the following coupled system:

$$\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,U) + B(U,u) + B(u,u) = \mathcal{B}_1\psi + \mathcal{B}_2\phi, \qquad -\frac{\mathrm{d}\tilde u}{\mathrm{d}t} + \nu\mathcal{A}^*\tilde u + B'(U+u)^*\tilde u = \mathcal{C}_1^*\mathcal{C}_1 u,\\
&u\in V,\quad \tilde u(t)\in V_r = \{v\in (H^1(\Omega))^3;\ \mathrm{div}\,v = 0 \text{ in } \Omega,\ v = \mathcal{C}_3^*\mathbf{r} \text{ on } \partial\Omega\},\quad t<T,\\
&u(0) = u_0 \quad \textit{and} \quad \tilde u(T) = \mathcal{C}_2^*\mathcal{C}_2\,u(T).
\end{aligned}\tag{3.21}$$

Proof. The existence of a robust control solution to the nonlinear problem was proved in Section 3.1 for the 3D case, subject to a small data constraint or a small T constraint, and in Section 3.2 for the 2D case, with no such constraint.

As in the linear case, the gradients $\mathcal{D}\mathcal{J}/\mathcal{D}\psi$ and $\mathcal{D}\mathcal{J}/\mathcal{D}\phi$ may be determined by computation of the coupled system (3.21), from whence (3.20) follows.

Uniqueness in the nonlinear case is proved by contradiction as follows: assume that $(\bar\psi,\bar\phi)$ and $(\tilde\psi,\tilde\phi)$ are two distinct saddle points in $\mathcal{X}\times\mathcal{Y}$. It follows from the statements of strict concavity (3.17) and strict convexity (3.19) that

$$\mathcal{J}(\tilde\psi,\tilde\phi) < \mathcal{J}(\tilde\psi,\bar\phi) < \mathcal{J}(\bar\psi,\bar\phi) \quad \text{and} \quad \mathcal{J}(\bar\psi,\bar\phi) < \mathcal{J}(\bar\psi,\tilde\phi) < \mathcal{J}(\tilde\psi,\tilde\phi).$$

This is a contradiction, and thus the saddle point $(\bar\psi,\bar\phi)$ is unique. □

Remark 3.10. *For the robust control problem on the bounded domain* $(\psi, \phi) \in \mathcal{X} \times \mathcal{Y}$ *as stated in Definition 3.1* (*and for which the gradients* $\mathcal{D}\mathcal{J}/\mathcal{D}\psi$ *and* $\mathcal{D}\mathcal{J}/\mathcal{D}\phi$ *are identified in Theorem 3.9*), *solutions* $(\bar{\psi}, \bar{\phi})$ *to the robust control problem may not necessarily satisfy* $(\mathcal{D}\mathcal{J}/\mathcal{D}\psi)(\bar{\psi}, \bar{\phi}) = (\mathcal{D}\mathcal{J}/\mathcal{D}\phi)(\bar{\psi}, \bar{\phi}) = 0$, *as they may be located on the boundary of the domain* $\mathcal{X} \times \mathcal{Y}$. *These equalities hold, however, if* $(\bar{\psi}, \bar{\phi})$ *is in the interior of* $\mathcal{X} \times \mathcal{Y}$ *and, in particular, if* $\mathcal{X}$ *and* $\mathcal{Y}$ *are all of* $L^2(0, T; L^2(\Omega)^d)$.

To summarize Section 3, for sufficiently large γ and l in both the 2D case and the 3D case (the latter of which is confined in the analysis either by a small data or a small T constraint), a solution to the robust control problem stated in Definition 3.1 exists and is unique. Further, the gradients $\mathcal{D}\mathcal{J}/\mathcal{D}\psi$ and $\mathcal{D}\mathcal{J}/\mathcal{D}\phi$, which may be used to determine this solution with a numerical algorithm such as that proposed in Section 5, may be identified as a simple function of an appropriately defined adjoint field.

4. Generalizations

We now consider two straightforward generalizations of the robust control framework laid out in Sections 2 and 3, first for the problem of boundary control (Section 4.1), then for the problem of data assimilation (Section 4.2).

4.1. The boundary control problem in a domain with corners

In this section, we will discuss the robust control problem assuming that the control ϕ now acts upon the flow by modification of the boundary conditions on the velocity u. It will be shown that the effect of boundary forcing on the flow velocity inside the domain Ω may be accounted for by a "lifting" procedure which constructs an equivalent interior forcing profile to account for the boundary forcing [22]. With such a construction, the proofs of existence and uniqueness of the robust control problem follow as in the previous sections with slight modifications. Further, the identification of the gradients $\mathcal{D}\mathcal{J}/\mathcal{D}\psi$ and $\mathcal{D}\mathcal{J}/\mathcal{D}\phi$ is, again, straightforward. In the present work we treat domains with corners, avoiding the smoothing of the boundary used in [1].

The cost functional considered in this section is analogous to that used in previous sections

$$\mathcal{J}(\psi, \phi) = \frac{1}{2}\int_0^T |\mathcal{C}_1 u|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}|\mathcal{C}_2 u(T)|^2_{L^2(\Omega)} - \int_0^T \left(\mathcal{C}_3 \nu \frac{\partial u}{\partial n}, \mathbf{r}\right)_{L^2(\partial\Omega)} \mathrm{d}t + \frac{1}{2}\int_0^T [l^2|\phi|^2_{L^2(\partial\Omega)} - \gamma^2|\psi|^2_{L^2(\Omega)}]\,\mathrm{d}t, \tag{4.1}$$

where the flow is governed by

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)U + (U\cdot\nabla)u + (u\cdot\nabla)u + \nabla p = B_1\psi, \quad \operatorname{div} u = 0, \quad u = \mathcal{B}_2\phi \ \text{ on } \partial\Omega, \qquad u = u_0 \ \text{ at } t = 0, \tag{4.2}$$

where we restrict $\phi \in H^1(0, T; (L^2(\partial\Omega))^3)$, $\mathcal{C}_3^*\mathbf{r}\cdot\mathbf{n} = 0$, and $\mathcal{B}_2$ to be a mapping from $(L^2(\partial\Omega))^3$ to $(H^{3/2+\epsilon}(\partial\Omega))^3$, $\epsilon > 0$, such that $(\mathcal{B}_2\phi, \mathbf{n})_{L^2(\partial\Omega)} = \int_{\partial\Omega}\mathcal{B}_2\phi\cdot\mathbf{n}\,\mathrm{d}\Gamma = 0$, where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$. Note the control forcing $\varphi = \mathcal{B}_2\phi$, which is allowed on all three velocity components, is confined to the boundary $\partial\Omega$ of the domain in the present section.

4.1.1. Transformation of problem to the interior forcing framework

For simplicity, let us assume a rectangular two or three-dimensional domain $\Omega = \prod_{i=1}^m(-L_i, L_i)$, where $m = 2$ or 3, as depicted in Fig. 4. Suppose that the boundary forcing belongs to the closure, in an appropriate Sobolev

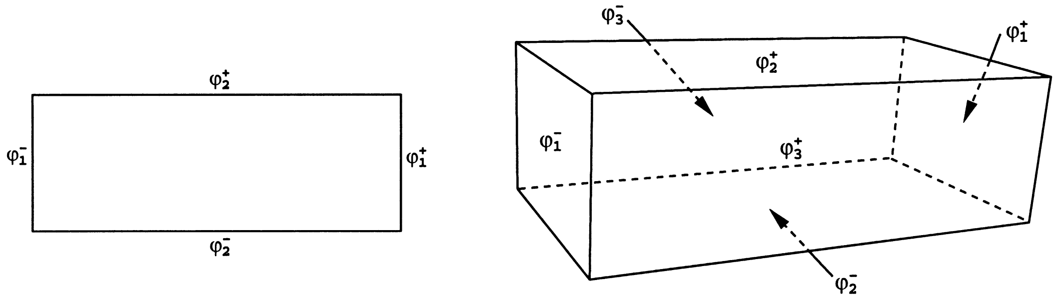

Fig. 4. Cornered domains considered for the problem of boundary forcing in Section 4.1.1. The analysis of the flow control problem in these 2D and 3D "driven cavities" is easily generalized to account for periodicity in one or two directions (i.e., channel flow and duct flow), and more generally, for domains of any convex shape.

space, of infinitely differentiable functions with compact support on each smooth component of $\partial\Omega$. Specifically, the boundary of Ω is given by

$$\partial\Omega=\bigcup_{i=1}^{n}(\Gamma_i^{+}\cup\Gamma_i^{-})\quad\text{and}\quad \Gamma_i^{\pm}=\{(x_1,\dots,x_n)\in\partial\Omega;\,x_i=\pm L_i\},$$

and the boundary controls $\varphi_i^{\pm}$ on each face $\Gamma_i^{\pm}$ of Ω satisfy

$$\varphi_i^{\pm}\in H^1(0,T;H_0^{3/2+\epsilon}(\Gamma_i^{\pm})),\quad\text{where }\ \epsilon>0.$$

Using [39], there exists $\Phi\in H^1(0,T;H^2(\Omega))$ such that

$$\frac{\partial\phi}{\partial t}-\nu\Delta\phi+\nabla\pi=0,\qquad \operatorname{div}\phi=0,\qquad \phi=\varphi_i^{\pm}\quad\text{on}\ \ \Gamma_i^{\pm},\qquad \phi=u_0\quad\text{at}\ \ t=0.$$

Furthermore, making use of Φ as an intermediate lifting, Hopf's technique [3] [22,30,36] may be used to determine a $\Theta\in H^1(0,T;H^2(\Omega))$ such that

$$\operatorname{div}\Theta=0,\qquad \Theta=\varphi_i^{\pm}\quad\text{on}\ \ \Gamma_i^{\pm},$$

where Θ is constructed such that

$$b(u,\Theta,u)\le\frac{\nu}{4}\|u\|^2. \tag{4.3}$$

Now set $v=u-\Theta$. As v, by construction, has homogeneous boundary conditions, we may return to the familiar abstract form

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\nu Av+B(v,U+\Theta)+B(U+\Theta,v)+B(v,v)=\mathcal{F},\quad v\in V,\quad v=u_0-\Theta(0)\ \text{ at }\ t=0, \tag{4.4}$$

where

$$\mathcal{F}=\mathcal{B}_1\psi-\frac{\mathrm{d}\Theta}{\mathrm{d}t}-\nu A\Theta-B(\Theta,\Theta)-B(U,\Theta)-B(\Theta,U).$$

[3] Note that the support of the lifting Θ of the boundary forcing φ determined by Hopf's approach is included in a neighborhood of the boundary. Note also that Hopf's technique can be easily generalized to work for the rectangular domains Ω of Fig. 4 and, in fact, for any convex domain.

Note that $\mathcal{F} \in L^2(0,T;H)$ and that

$$\|\mathcal{F}\|_{L^2(0,T;H)} \le C_0 \sum_{i=1}^{n} \|\varphi_i^{\pm}\|_{H^1(0,T;H_0^{3/2+\epsilon}(\Gamma_i^{\pm}))} + \|\mathcal{B}_1\psi\|_{L^2(0,T;H)}. \tag{4.5}$$

Theorem 4.1. *Existence and uniqueness of the solution to the robust control problems posed in Sections 2 and 3 extend directly to the case of boundary forcing.*

Proof. Noting the estimates (4.3) and (4.5), the existence and uniqueness of v follow promptly from the developments in Sections 2 and 3 applied to (4.4), with $U+\Theta$ replacing U and $\mathcal{F}$ replacing $\mathcal{P}f$, mutatis mutandis. The existence and uniqueness of $u = v + \Theta$ follow directly. □

4.1.2. Identification of gradients

The existence and uniqueness of a robust control solution for the problem with boundary control were proved in Section 4.1.1 by reducing it to the interior forcing problem, which has already been considered thoroughly in Sections 2 and 3. We now identify the gradients $\mathcal{DJ}/\mathcal{D}\psi$ and $\mathcal{DJ}/\mathcal{D}\phi$ necessary to find a solution to the robust data assimilation problem with the algorithm of Section 5 by coupling the flow system (4.2) with an adjoint system defined by

$$-\frac{\partial \tilde{u}}{\partial t} - \nu\Delta\tilde{u} + (\nabla[U+u])^{\mathrm{T}}\cdot\tilde{u} - (\nabla\tilde{u})\cdot[U+u] + \nabla\tilde{p} = \mathcal{C}_1^*\mathcal{C}_1 u, \quad \operatorname{div}\tilde{u} = 0, \quad \tilde{u} = \mathcal{C}_3^*\mathbf{r} \ \text{ on } \ \partial\Omega,$$
$$\tilde{u} = \mathcal{C}_2^*\mathcal{C}_2 u \ \text{ at } \ t = T. \tag{4.6}$$

Since

$$\begin{aligned}
&\int_0^T (\mathcal{C}_1^*\mathcal{C}_1 u, u')_{L^2(\Omega)}\,\mathrm{d}t + (\mathcal{C}_2^*\mathcal{C}_2 u(T), u'(T))_{L^2(\Omega)} - \int_0^T \left(\mathcal{C}_3 \nu \frac{\partial u'}{\partial n}, \mathbf{r}\right)_{L^2(\partial\Omega)} \mathrm{d}t \\
&\quad = \int_0^T \left(\left[-\frac{\partial\tilde{u}}{\partial t} - \nu\Delta\tilde{u} + (\nabla[U+u])^{\mathrm{T}}\cdot\tilde{u} - (\nabla\tilde{u})\cdot[U+u] + \nabla\tilde{p}\right], u'\right)_{L^2(\Omega)} \mathrm{d}t \\
&\qquad + (\tilde{u}(T), u'(T))_{L^2(\Omega)} - \int_0^T \left(\nu\frac{\partial u'}{\partial n}, \tilde{u}\right)_{L^2(\partial\Omega)} \mathrm{d}t \\
&\quad = \int_0^T \left(\tilde{u}, \left[\frac{\partial u'}{\partial t} - \nu\Delta u' + ([U+u]\cdot\nabla)u' + (u'\cdot\nabla)[U+u] + \nabla p'\right]\right)_{L^2(\Omega)} \mathrm{d}t \\
&\qquad + \int_0^T \Big[(\tilde{p}, u'\cdot\mathbf{n})_{L^2(\partial\Omega)} - (\tilde{p}, \nabla\cdot u')_{L^2(\Omega)} + (p', \nabla\cdot\tilde{u})_{L^2(\Omega)} - (p', \tilde{u}\cdot\mathbf{n})_{L^2(\partial\Omega)} \\
&\qquad - \left(\nu\frac{\partial\tilde{u}}{\partial n}, u'\right)_{L^2(\partial\Omega)} - ((\tilde{u}\cdot\mathbf{n})[U+u], u')_{L^2(\partial\Omega)}\Big]\,\mathrm{d}t \\
&\quad = \int_0^T (\tilde{u}, \mathcal{B}_1\psi')_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T \left(\left[-\nu\frac{\partial\tilde{u}}{\partial n} + \tilde{p}\mathbf{n}\right], \mathcal{B}_2\phi'\right)_{L^2(\partial\Omega)} \mathrm{d}t \\
&\quad = \int_0^T (\mathcal{B}_1^*\tilde{u}, \psi')_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T \left(\mathcal{B}_2^*\left[-\nu\frac{\partial\tilde{u}}{\partial n} + \tilde{p}\mathbf{n}\right], \phi'\right)_{L^2(\partial\Omega)} \mathrm{d}t,
\end{aligned}$$

it follows that

$$\frac{\mathcal{DJ}}{\mathcal{D}\psi}(\psi,\phi) = \mathcal{B}_1^*\tilde{u} - \gamma^2\psi \quad \text{and} \quad \frac{\mathcal{DJ}}{\mathcal{D}\phi}(\psi,\phi) = \mathcal{B}_2^*\left[-\nu\frac{\partial\tilde{u}}{\partial n} + \tilde{p}\mathbf{n}\right] + l^2\phi.$$

The appearance of $\tilde{p}$ in this derivation is due to the fact that we have allowed $\mathcal{B}_2\phi \cdot \mathbf{n} \neq 0$, i.e., we have allowed the case of boundary forcing by wall transpiration (blowing/suction). If we restrict the problem to wall-tangential control velocities only (i.e., $\mathcal{B}_2\phi \cdot \mathbf{n} = 0$), then the $\tilde{p}$ term will disappear from the expression for the gradient. Conversely, if we restrict the problem to wall-normal control velocities only, with $\mathcal{C}_3^*\mathbf{r}$ constant over the walls, then the $\partial\tilde{u}/\partial n$ term will disappear from the expression for the gradient.

To summarize Section 4.1, for sufficiently large γ, a solution to the robust control problem for the case of boundary forcing exists and is unique. Further, the gradients $\mathcal{DJ}/\mathcal{D}\psi$ and $\mathcal{DJ}/\mathcal{D}\phi$, which may be used to determine this solution with a numerical algorithm such as that proposed in Section 5, may be identified as a simple function of an appropriately defined adjoint field.

4.2. The data assimilation problem

In this section, we will discuss a robust estimation problem wherein the "control" to be determined is, in fact, the initial condition on the velocity field, i.e., $u_0 = \mathcal{B}_2\phi$. This framework is useful in data assimilation problems: given a set of measurements of some actual flow v on $[0, T]$, determine a "best" estimate as to the initial state u_0 in the model u that leads to the observed system behavior, while simultaneously forcing the model system with a small component of the worst-case disturbance ψ which perturbs u away from the observed system behavior. Chaotic problems, such as weather systems, are highly susceptible to the small disturbances present in all physical systems. Thus, this robust estimation framework should help to reduce the component of the initial state most susceptible to external disturbances and thereby prove to be a valuable tool for improving the fidelity of such estimates.

Define $w = u - v$ as the amount the estimated flow u differs from the actual flow v. A cost functional may be defined as in the previous problems, but now forced by the *measurement errors* $\mathcal{C}_1 w$, $\mathcal{C}_2 w(T)$, and $\mathcal{C}_3\nu(\partial w/\partial n)|_{\partial\Omega}$ on the interior, at the final time, and at the boundaries, respectively, such that

$$\mathcal{J}(\psi,\phi) = \frac{1}{2}\int_0^T |\mathcal{C}_1 w|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}|\mathcal{C}_2 w(T)|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^T \left|\mathcal{C}_3 \nu\frac{\partial w}{\partial n}\right|^2_{L^2(\partial\Omega)}\mathrm{d}t + \frac{l^2}{2}|\phi|^2_{L^2(\Omega)} - \frac{\gamma^2}{2}\int_0^T |\psi|^2_{L^2(\Omega)}\,\mathrm{d}t, \tag{4.7}$$

where $\mathcal{C}_3^*\mathcal{C}_3\nu(\partial w/\partial n)\cdot\mathbf{n} = 0$. The measurements of the actual flow $\mathcal{C}_1 v$, $\mathcal{C}_2 v(T)$, and $\mathcal{C}_3\nu(\partial v/\partial n)|_{\partial\Omega}$ are assumed to be given. In order to find the best estimate u of the actual flow v, we seek the best initial conditions ϕ, subject to the worst disturbance forcing ψ, such that $\mathcal{J}$ is minimized, where the estimate u is governed by

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + B(u,U) + B(U,u) + B(u,u) = \mathcal{B}_1\psi, \quad u \in V, \quad u = \mathcal{B}_2\phi \text{ at } t = 0, \tag{4.8}$$

where $\phi \in (L^2(\Omega))^3$ and $\mathcal{B}_2$ is a mapping from $(L^2(\Omega))^3$ to V.

Theorem 4.2. *Existence and uniqueness of the solution to the robust control problems posed in Sections 2 and 3 extend directly to the data assimilation framework.*

Proof. The development of Sections 2 and 3 extend directly to the present case with no further estimates required. □

4.2.1. Identification of gradients

The existence and uniqueness of a robust solution for the data assimilation problem were found by appealing directly to the interior forcing problem, which has already been considered thoroughly in Sections 2 and 3. We now identify the gradients $\mathcal{D}\mathcal{J}/\mathcal{D}\psi$ and $\mathcal{D}\mathcal{J}/\mathcal{D}\phi$ necessary to find a solution to the robust data assimilation problem with the algorithm of Section 5. The derivation is very similar to those encountered earlier.

Differentiation of (4.7) leads to expressions for the gradients in weak form:

$$\begin{aligned}\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\psi}(\bar{\psi},\bar{\phi})\cdot\psi' &= \int_0^T \left(\mathcal{C}_1 w, \mathcal{C}_1\frac{\mathcal{D}u}{\mathcal{D}\psi}\cdot\psi'\right)_{L^2(\Omega)} \mathrm{d}t + \left(\mathcal{C}_2 w(T), \mathcal{C}_2\frac{\mathcal{D}u(T)}{\mathcal{D}\psi}\cdot\psi'\right)_{L^2(\Omega)} \\ &\quad + \int_0^T \left(\mathcal{C}_3\nu\frac{\partial w}{\partial n}, \mathcal{C}_3\nu\frac{\partial}{\partial n}\frac{\mathcal{D}u}{\mathcal{D}\psi}\cdot\psi'\right)_{L^2(\partial\Omega)} \mathrm{d}t - \gamma^2\int_0^T(\bar{\psi},\psi')_{L^2(\Omega)}\,\mathrm{d}t,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi}(\bar{\psi},\bar{\phi})\cdot\phi' &= \int_0^T \left(\mathcal{C}_1 w, \mathcal{C}_1\frac{\mathcal{D}u}{\mathcal{D}\phi}\cdot\phi'\right)_{L^2(\Omega)} \mathrm{d}t + \left(\mathcal{C}_2 w(T), \mathcal{C}_2\frac{\mathcal{D}u(T)}{\mathcal{D}\phi}\cdot\phi'\right)_{L^2(\Omega)} \\ &\quad + \int_0^T \left(\mathcal{C}_3\nu\frac{\partial w}{\partial n}, \mathcal{C}_3\nu\frac{\partial}{\partial n}\frac{\mathcal{D}u}{\mathcal{D}\phi}\cdot\phi'\right)_{L^2(\partial\Omega)} \mathrm{d}t + l^2(\bar{\phi},\phi')_{L^2(\Omega)}.\end{aligned}$$

In order to determine a solution to the robust data assimilation problem, we define an adjoint state by the equation

$$\begin{aligned}&-\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} + \nu\mathcal{A}^*\tilde{u} + B'(U+u)^*\tilde{u} = \mathcal{C}_1^*\mathcal{C}_1 w,\\ &\tilde{u}(t)\in V_w = \left\{v\in(H^1(\Omega))^3;\ \mathrm{div}\,v=0 \text{ in } \Omega,\ v=-\mathcal{C}_3^*\mathcal{C}_3\nu\frac{\partial w}{\partial n} \text{ on } \partial\Omega\right\},\\ &t<T,\quad \tilde{u}=\mathcal{C}_2^*\mathcal{C}_2\,w\in H \quad \text{at } t=T,\end{aligned}$$

where $\mathcal{A}^*$ is defined by

$$(u',\mathcal{A}^*\tilde{u})_{L^2(\Omega)} = (Au',\tilde{u})_{L^2(\Omega)} + \left(\frac{\partial u'}{\partial n},\tilde{u}\right)_{L^2(\partial\Omega)} \qquad \text{for } u'\in D(A) \text{ and } \tilde{u}\in V_w.$$

Note again that $\mathcal{C}_3^*\mathcal{C}_3\nu(\partial w/\partial n)\cdot\mathbf{n}=0$. Note that the adjoint state is forced by the measurement errors $\mathcal{C}_1 w, \mathcal{C}_2 w(T)$, and $\mathcal{C}_3\nu(\partial w/\partial n)|_{\partial\Omega}$. From integration by parts and the regularity of u, u' and $\tilde{u}$, which follows as in the previous sections, we have

$$\begin{aligned}&\int_0^T(\mathcal{C}_1^*\mathcal{C}_1 w,u')_{L^2(\Omega)}\,\mathrm{d}t + (\mathcal{C}_2^*\mathcal{C}_2 w(T),u'(T))_{L^2(\Omega)} + \int_0^T\left(\mathcal{C}_3^*\mathcal{C}_3\nu\frac{\partial w}{\partial n},\nu\frac{\partial u'}{\partial n}\right)_{L^2(\partial\Omega)}\mathrm{d}t\\ &\quad = \int_0^T\left(\left[-\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t}+\nu\mathcal{A}^*\tilde{u}+B'(U+u)^*\tilde{u}\right],u'\right)_{L^2(\Omega)}\mathrm{d}t\\ &\qquad +(\tilde{u}(T),u'(T))_{L^2(\Omega)} - \int_0^T\left(\tilde{u},\nu\frac{\partial u'}{\partial n}\right)_{L^2(\partial\Omega)}\mathrm{d}t\\ &\quad = \int_0^T\left(\tilde{u},\left[\frac{\mathrm{d}u'}{\mathrm{d}t}+\nu Au'+B'(U+u)u'\right]\right)_{L^2(\Omega)}\mathrm{d}t + (\tilde{u}(0),u'(0))_{L^2(\Omega)}\\ &\quad = \int_0^T(\tilde{u},\mathcal{B}_1\psi')_{L^2(\Omega)}\,\mathrm{d}t + (\tilde{u}(0),\mathcal{B}_2\phi')_{L^2(\Omega)}\\ &\quad = \int_0^T(\mathcal{B}_1^*\tilde{u},\psi')_{L^2(\Omega)}\,\mathrm{d}t + (\mathcal{B}_2^*\tilde{u}(0),\phi')_{L^2(\Omega)}.\end{aligned}$$

Thus, (4.9), with $\phi' = 0$ and taking $\psi' \in L^2(0, T; L^2(\Omega)^d)$ as arbitrary, leads to an expression for the gradient $\mathcal{DJ}/\mathcal{D}\psi$:

$$\frac{\mathcal{DJ}}{\mathcal{D}\psi}(\bar{\psi}, \bar{\phi}) = \mathcal{B}_1^* \tilde{u} - \gamma^2 \psi.$$

Similarly, with $\psi' = 0$ and taking $\phi' \in L^2$ as arbitrary,

$$\frac{\mathcal{DJ}}{\mathcal{D}\phi}(\bar{\psi}, \bar{\phi}) = \mathcal{B}_2^* \tilde{u}(0) + l^2 \phi.$$

To summarize Section 4.2, for sufficiently large γ and l, a solution to the robust setting of the data assimilation problem exists and is unique. Further, the gradients $\mathcal{DJ}/\mathcal{D}\psi$ and $\mathcal{DJ}/\mathcal{D}\phi$, which may be used to determine this solution with a numerical algorithm such as that proposed in Section 5, may be identified as a simple function of an appropriately defined adjoint field.

5. Numerical algorithm for determination of robust control solution

A selected number of important linear robust control problems may be derived from the Navier–Stokes equation and discretized accurately with a system of small state dimension [$N < \mathrm{O}(100)$]. For such problems, the two-point boundary-value problem of (2.22) may be stated and solved as a Riccati problem, as demonstrated by Bewley and Liu [8] for the problem of stabilization of plane channel flow. For nonlinear robust control problems of fairly low state dimension, the two-point boundary-valued problem may be stated as the Hamilton–Jacobi–Bellman inequality and solved via the notions of L^2 gain, passivity, and control Lyapunov functions, as described by Isidori [23], van der Schaft [33] and Freeman and Kokotovic [17].

The majority of linear and nonlinear problems in fluid mechanics, however, require quite a large state dimension for adequate resolution [$N > \mathrm{O}(10^5)$]. For such problems, a computational approach which does not rely on the computation and storage of $\mathrm{O}(N^2)$ fields is an absolute necessity. As suggested for the optimal case by Abergel and Temam [1], an iterative numerical algorithm is now proposed to find a saddle-point solution to the two-point boundary value problem of both linear and nonlinear robust control problems based on the repeated computation of an $\mathrm{O}(N)$ adjoint field.

Algorithm 1.

1. *Initialize* $k = 0$ *and* $(\psi^0, \phi^0) = 0$ *on* $t \in [0, T]$, *where* k *is the iteration index and* (ψ^k, ϕ^k) *is the numerical approximation of the disturbance and the control during the kth iteration of the algorithm.*
2. *Determine the state* u^k *on* $[0, T]$ *from the state equation (Navier–Stokes) based on the initial conditions* u_0 *and with the forcing* (ψ^k, ϕ^k).
3. *Determine the adjoint* $\tilde{u}^k$ *on* $[0, T]$ *from the adjoint equation based on the state* u^k.
4. *Determine local expressions for the gradients*

$$\frac{\mathcal{DJ}}{\mathcal{D}\psi}(\psi^k, \phi^k) \quad \textit{and} \quad \frac{\mathcal{DJ}}{\mathcal{D}\phi}(\psi^k, \phi^k)$$

   based on the adjoint $\tilde{u}^k$.
5. *Determine the updated disturbance* ψ^{k+1} *with*

$$\psi^{k+1} = \psi^k + \alpha^k \frac{\mathcal{DJ}}{\mathcal{D}\psi}(\psi^k, \phi^k),$$

   where $0 < C_1 \le \alpha^k \le C_2 < 1$, *where* C_1 *and* C_2 *depend on the second derivative of* $\mathcal{J}$.

6. *Determine the updated control* ϕ^{k+1} *with*

$$\phi^{k+1} = \phi^k - \beta^k \frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi}(\psi^k, \phi^k),$$

where $0 < C_1 \leq \beta^k \leq C_2 < 1$.

7. *Increment index*: $k = k + 1$. *Repeat from step* 2 *until converged*.

The proof of the convergence of Algorithm 1 is currently under development and will appear elsewhere.

In order to arrive at an algorithm which is numerically efficient, but for which a proof of convergence is not available, α^k and β^k themselves may be determined by an iterative procedure, specifically as given below.

Algorithm 2. *Follow the same procedure as in Algorithm 1, but now*:

- *Let* α^k *be selected by a numerically robust line maximization algorithm, based on repeated "trial" computations of* ψ^{k+1} *for various different choices of* α^k *and computing the resulting effect on the cost functional* $\mathcal{J}(\psi^{k+1}, \phi^k)$. *Such an approach guarantees local line maximization of* $\mathcal{J}$ *in the direction of the gradient* $[\mathcal{D}\mathcal{J}/\mathcal{D}\psi](\psi^k, \phi^k)$ *from the point* (ψ^k, ϕ^k) *even for nonlinear problems. Efficient numerical algorithms for such a line maximization* (*i.e., optimization of the single scalar parameter* α^k) *are well established. Note that* α^k *is chosen while holding the control* ϕ^k *fixed*.
- *Let* β^k *be selected by a line minimization algorithm, based on repeated "trial" computations of* ϕ^{k+1} *for various different choices of* β^k *and computing the resulting effect on the cost functional* $\mathcal{J}(\psi^k, \phi^{k+1})$, *in a manner analogous to that for* α^k. *Note that* β^k *is chosen while holding the disturbance* ψ^k *fixed*.

Note that Algorithm 2 may be modified by setting $\alpha^k = 0$ (i.e., $\psi^{k+1} = \psi^k$) for k odd and setting $\beta^k = 0$ (i.e., $\phi^{k+1} = \phi^k$) for k even. Note also that, for the purpose of numerical computation, it is often most efficient to compute the line minimizations only approximately, in order to reduce the number of computations required to select α^k and β^k.

6. Conclusions

A framework for robust control has been developed for problems governed by the Navier–Stokes equation. Existence and uniqueness of the solution to the robust control problem have been proven, and upper bounds on the minimum value of γ_0 for which a solution exists have been established. Cost functionals which account for both the regulation and the terminal control of both interior quantities and boundary quantities have been accounted for. Interior forcing, boundary forcing, and the optimization of the initial state (i.e., data assimilation) have been considered. Together, this set of problems constitutes a complete family of problems governed by the flow/adjoint two-point boundary value problem. Finally, a tractable numerical algorithm (based on repeated computations of an adjoint field) to solve the robust control problem has been proposed.

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