Exponential stabilization of a variable-length pendulum with nonlinear feedback and a curious caveat

Thomas Bewley, *UC San Diego* and *NASA Jet Propulsion Lab*

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Abstract

This paper characterizes a C^∞ nonlinear feedback control strategy that exponentially stabilizes the oscillations of the angle $\phi(t)$ of a classical variable-length pendulum. This simple system, once properly nondimensionalized, is governed by $\ddot{\phi} = -(\sin\phi + 2\,\dot{\ell}\,\dot{\phi})/\ell$. The nonlinear feedback control rule applied in this work takes the nondimensionalized length of the pendulum as $\ell(t) = 1 + \delta(t)\,\phi(t)\,\dot{\phi}(t)$, with a feedback gain $\delta(t)$ that is gradually increased as convergence is approached. This is achieved by taking $\delta(t) = C/V_s(t)$, where $V_s(t)$ denotes a simplified measure of the energy of the pendulum oscillations, $V_s(t) = \dot{\phi}(t)^2/2 + 1 - \cos\,\phi(t)$. Convergence is attained in the sense of Lyapunov, in the discrete-time setting introduced by Karafyllis (2012), integrating the derivative of $V_s(t)$ along system trajectories, 2π radians around the origin in phase space at a time. The fractional rate of convergence, f, after each full rotation of the system trajectory about the origin in phase space, for small oscillations of $\phi(t)$, is determined numerically and written, for finite C, as $\Delta V_s/V_s = -Cp = -f$; for $0 < C \le 0.5$, it works out that $3\pi > p > 2$ (that is, p is nearly constant over a very large range of C), with $p \to 3\pi$ as $C \to 0$. Conveniently, it follows for small V_s that $C = \max_t |\ell(t) - 1|$ for each oscillation of the system; exponential stabilization of the oscillations of $\phi(t)$ is thus seen to be achieved with this approach by maintaining the oscillations of $\ell(t)$ at a constant amplitude (set by C) as long as the control is applied, not by reducing the amplitude of the oscillations of $\ell(t)$ as the oscillations of $\phi(t)$ subside.

1 Introduction

The study of the dynamics of fixed-length pendulums dates back to Galileo Galilei and Christiaan Huygens in the 1600s. The study of the dynamics of variable-length pendulums, especially those of periodically-varying length $\ell(t)$, is also classical (see, e.g., [1, 2]). The stabilization of a fixed-length inverted pendulum via open-loop vertical vibrations of its pivot point is a well-known related problem [3]. The feedback control problem of pumping up the oscillations of the angle $\phi(t)$ of a variable-length pendulum, via impulsive changes of $\ell(t)$ appropriately synchronized with the oscillations of $\phi(t)$, has also enjoyed some attention (see [4, 5]). This prior work reveals that (a) effective feedback controls for stabilizing the oscillations of a variable-length pendulum require an oscillation of $\ell(t)$ at *twice* the fundamental frequency of oscillation of $\phi(t)$, and (b) the oscillations of $\ell(t)$ are required to be phased such that $\ell(t)$ reaches its maximums during the *upswings* of the pendulum [when $|\phi(t)|$ is increasing], and $\ell(t)$ reaches it minimums during the *downswings* of the pendulum [when $|\phi(t)|$ is decreasing]. These two requirements motivate the simple feedback rule $\ell = 1 + \delta\,\phi\,\dot{\phi}$ proposed and studied in the present paper, which is evidently the simplest, smoothest (C^∞) feedback control rule available which achieves both of these requirements (note that a linear controller is insufficient to provide the required frequency doubling). The convergence behavior of this new feedback control strategy is analyzed in the present work.

Denoting with "prime" differentiation w.r.t. (dimensional) time τ [e.g., $\Phi' = d\Phi/d\tau$], and following, e.g., [5], the full nonlinear equation of motion (EOM) of a variable-length hanging simple pendulum of length $L(\tau) > 0$, angle $\Phi(\tau)$, and (constant) mass $m > 0$, rotating in a plane about a fixed support point O, may be found simply via conservation of angular momentum $H' = N$, where $H = mL^2\Phi'$ is the angular momentum of the body about O, and $N = -mgL\sin\Phi$ is the net torque about O due to the effective acceleration of gravity $g > 0$. Neglecting the effects of the displacement and drag of the air that the pendulum moves through, as well as the friction of the rotation about the pivot point, we thus obtain

$$\Phi'' = -(g\sin\Phi + 2\,\Phi'\,L')/L.$$

Defining $\tau_0 = \sqrt{L_0/g}$, where L_0 is the nominal length of the pendulum, simple rescaling with $t = \tau/\tau_0$ and $\ell = L/L_0$, and writing $\phi(t) = \Phi(\tau)$, leads to the equivalent nondimensional form

$$\boxed{\ddot{\phi} = -(\sin\phi + 2\,\dot{\phi}\,\dot{\ell})/\ell \quad \Leftrightarrow \quad \ell\,\ddot{\phi} + \sin\phi = -2\,\dot{\phi}\,\dot{\ell}}\,, \tag{1}$$

where "dot" denotes differentiation w.r.t. (nondimensional) time t [e.g., $\dot\phi = d\phi/dt$], and where, in the controlled setting, $\ell(t)$ is taken as the control variable. For lack of a better short name, we will refer to (1) as the varpend (variable-length pendulum) oscillator, which is seen to be of similar simplicity, and perhaps even more important physical and engineering significance, than both the van der Pol oscillator $\ddot\phi - \mu(1-\phi^2)\dot\phi + \phi = 0$, and the Duffing oscillator $\ddot\phi + \delta\dot\phi + \alpha\phi + \beta\phi^3 = 0$, often studied in introductory nonlinear systems courses.

Accounting for the displacement of the air that the pendulum moves through using an "added mass" formulation, as proposed by [7], reduces the effective gravity g, but leaves the essential form of (1) unchanged. Using such an added mass formulation, the varpend oscillator (1), with additional RHS damping terms to also account for air drag, provides a useful model of the oscillations of a singly-tethered hot-air (or, helium-filled) balloon with substantial excess lift (and, thus, a taut tether). Such applications usually have a winch already installed, to launch and land the tethered balloon; this existing winch can be used to modulate $\ell(t)$, implementing the control strategy proposed herein. This engineering application is important because wind can often introduce hazardous oscillations of such balloons [8], forcing the system via alternate vortex shedding at frequencies near the resonant frequency of the varpend oscillator itself, akin to the famous Tacoma Narrows Bridge disaster (see, e.g., [9]); the present control strategy can be used to subdue such sympathetic oscillations before they get too large.

The varpend oscillator (1) may also work well, in certain settings, for modelling the oscillations of tethered satellite systems [10], which are also already equipped with winches to deploy and retrieve the tethered satellite subsystem; implementing the controls developed herein in order to subdue the relative oscillations of such tethered satellites, using battery power only (recharged by solar panels) without expending any propellant (which is a valuable limited resource in orbit), might also be found to be useful.

The total (kinetic + potential) energy of the oscillations of this system may be written in the nondimensional setting as

$$V = (\ell^2\,\dot\phi^2 + \dot\ell^2)/2 + \ell\,(1-\cos\phi) \geq 0. \tag{2}$$

Note that V approaches zero as the oscillations of the pendulum angle subside, and if

$$V < 2, \tag{A0}$$

the oscillations of the uncontrolled ($\ell = 1$) system remain within $-\pi < \phi(t) < \pi$. For convenience, will will thus assume (A0) in the remainder of this paper.

In the uncontrolled case, with $\ell = 1$, it follows that $V \leq (\dot\phi^2 + \phi^2)/2$, and that

$$dV/dt = \dot\phi\,(\ddot\phi + \sin\phi) = 0.$$

For $\ell = 1$ and small deflections of ϕ (keeping only the linear term in the expansion of $\sin\phi$), (1) reduces to the EOM of a simple harmonic oscillator, $\ddot\phi + \phi = 0$, with solution $\phi(t) = c_1 \sin(\phi + c_2)$; keeping only the linear and cubic terms in the expansion of $\sin\phi$ (1) reduces it to a Duffing oscillator, $\ddot\phi + \phi - \phi^3/6 = 0$ (that is, an oscillator with a nonlinear spring, which *softens* for finite-amplitude oscillations, thus resulting in a somewhat longer oscillation period than for infinitesimal oscillations). If $\ell = 1$ but the oscillations of ϕ are not assumed to be small, the full solution of (1) may be written in terms of the Jacobi amplitude function $\mathrm{am}(\cdot,\cdot)$ as

$$\phi(t) = 2\,\mathrm{am}\left(\sqrt{(c_1+2)(t+c_2)^2}/2,\, 4/(c_1+2)\right), \tag{3}$$

where again $\{c_1, c_2\}$ are functions of of the initial conditions on $\{\phi, \dot\phi\}$. This solution is periodic, with period T_0 satisfying $2\pi < T_0 < \infty$ for $0 < V < 2$ [see (A0)], and amplitude $|\phi|_{\max} = \mathrm{acos}(1-V) < \pi$. For small V, $\phi(t)$ by this solution is again nearly sinusoidal; for larger V, the peaks of $\phi(t)$ become flattened, and the oscillation period increases. Of course, the Jacobi amplitude function is not exactly computable in terms of a finite number of elementary operations, and in practice must be determined via integration. The "closed form" solution (3) is thus, perhaps, of rather limited practical utility. Note instead that the scalar second-order ODE (1), which is not at all stiff, is quite simple to integrate numerically using, e.g., a standard RK4 method, with a small timestep Δt. This may be done with very high accuracy, and can easily be extended to the controlled settings discussed below.

1.1 Feedback control rules tested

The simple feedback control strategy proposed and considered in this work is to apply

$$\ell = 1 + \delta\,\phi\,\dot\phi \tag{4}$$

to the varpend oscillator (1) discussed above.

Two specific cases of interest for the tuning of δ are considered. In §2, we consider the simplest case, with

$$\delta = \text{constant} > 0. \tag{5a}$$

Unfortunately, this feedback control approach is found (in §2) to provide only asymptotoic convergence as, by the control rule (4) in this constant δ case, the amplitude of oscillations of the pendulum length $\ell(t)$ diminish to zero faster than the amplitude of oscillations of $\phi(t)$ themselves diminish to zero.

We thus consider, in §3, a more aggressive control approach, which scales the feedback gain $\delta(t)$ with the inverse of a measure of the energy of the oscillations of $\phi(t)$. This case may be written as

$$\delta = C/V_s \quad \text{where} \quad V_s = \dot{\phi}^2/2 + 1 - \cos\phi \geq 0. \tag{5b}$$

Note that $V_s(t)$ is a simplified approximation of the energy of the oscillations of the system given in full by $V(t)$ in (2). [The use of more accurate approximations of $V(t)$ in the computation of $\delta(t)$ were also tested, but did not provide any apparent benefit in the resulting system behavior.] For small C, $V_s(t)$ [and, thus, the feedback gain $\delta(t)$] is found to change only gradually over time when implementing this approach. The synchronization of the essential oscillations of $\ell(t)$ with $\phi(t)$ and $\dot{\phi}(t)$ (that is, with $\ell(t)$ oscillating at the second-harmonic of the oscillations of $\phi(t)$, with the appropriate phase; see §1), by applying (4) with (5b), is thus largely the same as in the constant δ case [applying (4) with (5a)], with the feedback gain $\delta(t)$ now growing steadily as convergence is approached.

Note also that, from here forward, for notational simplicity, we will largely drop the (t) indications, used sporadically above, from the notation used.

1.2 Numerical results

To better illustrate the behavior of the controlled varpend oscillator (1) studied in this work, we take the (perhaps, unusual) first step of introducing the dynamics of this controlled system with a couple of numerical simulations (using standard RK4 with $\Delta t = 0.01$, and ICs of $\phi(0) = 1$ and $\phi'(0) = 0$). The dot-dashed curves in Figure 1 illustrate the behavior of this system when the control (4) with (5a) is applied, taking $\delta = 0.2$, and the solid curves in Figure 1 illustrate the behavior of this system when the control (4) with (5b) is applied, taking $C = 0.1$. Asymptotic convergence is seen in the former case, and exponential convergence (over 13 orders of magnitude, down to machine zero) is seen in the latter case; this paper sets out to explain this behavior.

2 Case with constant δ

In the simple case with constant δ, it follows from (4) with (5a) that

$$\dot{\ell} = \delta\,(\dot{\phi}^2 + \phi\,\ddot{\phi}). \tag{6}$$

Given (1), (4), and (6), noting that $\dfrac{a}{1+b} = a - \dfrac{b\,a}{1+b}$, it follows that

$$\begin{aligned}
(1 + \delta\,\phi\,\dot{\phi})\,\ddot{\phi} + \sin\phi &= -2\,\dot{\phi}\,\delta\,(\dot{\phi}^2 + \phi\,\ddot{\phi}),\\
(1 + 3\,\delta\,\phi\,\dot{\phi})\,\ddot{\phi} + \sin\phi &= -2\,\delta\,\dot{\phi}^3,\\
\ddot{\phi} + \frac{\sin\phi}{1 + 3\,\delta\,\phi\,\dot{\phi}} &= \frac{-2\,\delta\,\dot{\phi}^3}{1 + 3\,\delta\,\phi\,\dot{\phi}},\\
\ddot{\phi} + \sin\phi &= \frac{-2\,\delta\,\dot{\phi}^3 + 3\,\delta\,\phi\,\dot{\phi}\,\sin\phi}{1 + 3\,\delta\,\phi\,\dot{\phi}}.
\end{aligned} \tag{7}$$

We must ensure that the denominator on the RHS of (7) remains positive, and thus that $\ddot{\phi}$ remains bounded in this model. This may be accomplished easily by leveraging the Cauchy-Schwarz inequality, in the form $|\phi\,\dot{\phi}| \leq (\phi^2 + \dot{\phi}^2)/2$, from which it follows immediately that $\ddot{\phi}$ remains bounded by assuming that

$$0 < \delta\,(\phi^2 + \dot{\phi}^2) < 2/3; \tag{A1}$$

that is, for a given maximum value of $(\phi^2 + \dot{\phi}^2)$ selecting δ sufficiently small, or for a given δ selecting $(\phi^2 + \dot{\phi}^2)$ sufficiently small. This upper bound on the maximum allowable δ is tight (that is, non-conservative); for $\delta\,(\phi^2 + \dot{\phi}^2) = 2/3$, when $\dot{\phi} = -\phi$, the denominator of (7) goes to zero, and $\ddot{\phi}$ diverges. For $\delta\,(\phi^2 + \dot{\phi}^2) < 2/3$, no combinations of

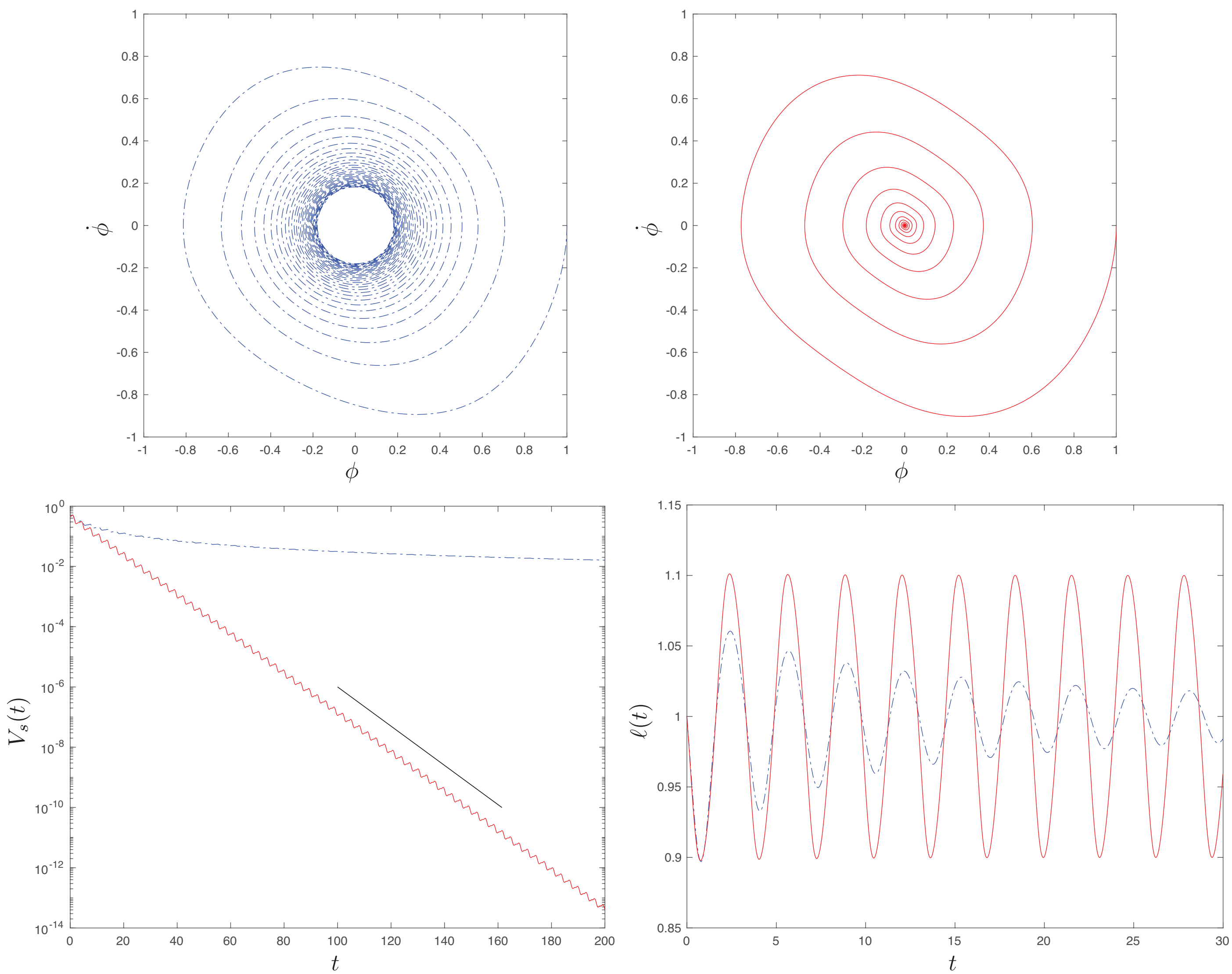

Figure 1: Simulations of the controlled varpend oscillator: (a,b) trajectories in phase space $\{\phi, \dot{\phi}\}$, (c) a simplified measure of the energy of the oscillations of $\phi(t)$ as a function of time, $V_s(t)$, (d) pendulum length as a function of time, $\ell(t)$. The dot-dashed curves in (a,c,d) illustrate the behavior of (4) with (5a), taking $\delta = 0.2$, and the solid curves in (b,c,d) illustrate the behavior of (4) with (5b), taking $C = 0.1$. For reference, the black line in (c) indicates a slope corresponding to a reduction of $V_s(t)$ by an order of magnitude every $\Delta t = 2.445 \cdot 2\pi$ nondimensional time units, as predicted by the analysis in the last paragraph of §3.2.

ϕ and $\dot\phi$ drive the denominator to zero. In practice, δ should be kept significantly smaller than the limit implied by this upper bound in order to ensure a well-behaved controlled system, accurately governed by the model given in (1) with appropriately small accelerations $\ddot\phi$.

Equation (7) is the formula for a "damped" oscillator; whether the (cubic) RHS terms result, in total, in stabilization or destabilization of this oscillator is the central question of interest addressed below.

Noting (7) and the definition of V_s in (5b), it follows that

$$\frac{dV_s}{dt} = \dot\phi\,(\ddot\phi + \sin\phi) = \dot\phi\left(\frac{-2\,\delta\,\dot\phi^3 + 3\,\delta\,\phi\,\dot\phi\,\sin\phi}{1 + 3\,\delta\,\phi\,\dot\phi}\right) = \frac{-2\,\delta\,\dot\phi^4 + 3\,\delta\,\phi\,\dot\phi^2\,\sin\phi}{1 + 3\,\delta\,\phi\,\dot\phi}. \tag{8}$$

It is seen in the (exact) expression above that, if $\delta = 0$, then $dV_s/dt = 0$, and thus V_s is a constant, set by the initial conditions on ϕ and $\dot\phi$. It follows from (8) that

$$\frac{dV_s}{dt} \le -\frac{2\,\delta\,\dot\phi^4}{1 + 3\,\delta\,\phi\,\dot\phi} + \frac{3\,\delta\,\phi^2\,\dot\phi^2}{1 + 3\,\delta\,\phi\,\dot\phi}. \tag{9}$$

Note that, for small ϕ (and thus $\sin\phi \approx \phi$), the bound given in (9) is tight. It is seen above that, assuming (A1), the time derivative of V_s along system trajectories, considered now as a candidate Lyapunov function, is bounded above by two terms, forming a "competition" of sorts. [Quantification of, in total, which term eventually "wins" this competition is the key question of interest.] The first term on the RHS is negative semidefinite (that is, stabilizing), and goes to zero (quartically) whenever $\dot\phi$ goes to zero, whereas the second term is positive semidefinite (that is, destabilizing), and goes to zero (quadratically) whenever either ϕ or $\dot\phi$ goes to zero. Whenever $\phi^2 > (2/3)\dot\phi^2$ (which happens, for two finite periods during each revolution of the system trajectory around the origin in phase space, even for infinitesimal oscillations), the magnitude of the destabilizing term becomes greater than the magnitude of the stabilizing term, and thus we can *not* establish that $dV_s/dt \le 0$ everywhere (thus proving traditional Lyapunov stability) in *any* finite region surrounding the origin using this candidate Lyapunov function.

Further, no better candidate Lyapunov function for establishing convergence of this controlled system has yet been discovered (though, not for lack of trying). In this regard, possibly taking the candidate Lyapunov function V_s considered above as an initial guess, a constructive or "learning" strategy, such as that proposed in [11], may well be able to identify a new Lyapunov function $V_L \ge 0$, with $dV_L/dt < 0$ along system trajectories everywhere within a finite region surrounding the origin in phase space. However, our efforts at generating such a proof of stability, using such a numerically-generated Lyapunov function V_L, have so far been unsuccessful.

As mentioned in (3), taking $\delta = 0$ in (7), the solution of the resulting ODE, $\ddot\phi + \sin\phi = 0$, in fact may be written in "closed-form" in terms of the Jacobi amplitude function; this solution reduces to simple sinusoidal oscillations (that is, to circular trajectories in phase space $\{\phi, \dot\phi\}$) if V_s is taken as small. On the other hand, taking $\delta > 0$, a similar "closed-form" solution of (7), with its full (rather complicated) RHS, appears to be unavailable.

However, for finite $\delta > 0$, assuming (A1) [but, *not* assuming infinitesimal V_s], the (oscillatory) dynamics of (7) are quite easy to simulate numerically with high accuracy [the ODE (7) is not stiff], using a standard RK4 method and small timestep Δt. A typical such simulation result, taking $\{\phi, \dot\phi\}_{t=0} = \{1, 0\}$ and $\delta = 0.2$ [thus satisfying (A1)], is illustrated in Figure 1; other ICs and values of δ satisfying (A1) lead, qualitatively, to the same simple behavior. It is observed that, for finite δ, oscillations do initially decay, but that this decay rate diminishes rapidly as V_s gets small, with the system oscillations eventually approaching nearly circular trajectories in phase space.

This behavior may be understood by considering (7) for finite $\delta > 0$ but small V_s. Defining

$$\epsilon = \sqrt{2\,V_s} \approx (\phi^2 + \dot\phi^2)^{1/2}, \tag{10a}$$

the effect of both terms on the RHS of (7) in this setting are diminished [that is, they are $O(\epsilon^3)$], and $\sin\phi = \phi + O(\epsilon^3)$. Thus, again, the system trajectories $\{\phi, \dot\phi\}$ are nearly circular, and may be well approximated as $\{\varphi, \dot\varphi\}$ where

$$\varphi = \epsilon\,\sin(t + \psi), \quad \dot\varphi = \epsilon\,\cos(t + \psi). \tag{10b}$$

To establish that the net effect of the stabilizing term is greater than that of the destabilizing term on the RHS of (9) in the limit of small ϵ, thus reducing V_s with each revolution around the origin in phase space, we may simply integrate the RHS of (9) [dropping the $O(\epsilon^2)$ terms added to unity in the denominator, which in the small ϵ limit are negligible] around 2π radians of the approximate trajectory $\{\varphi, \dot\varphi\}$ suggested in (10), which gives

$$\begin{aligned}\Delta V_s = \delta \int_0^{2\pi} \left[-2\,\dot\varphi^4 + 3\,\varphi^2\,\dot\varphi^2\right]\,dt &= \epsilon^4\,\delta \int_0^{2\pi} \left[-2\,(\cos(t+\psi))^4 + 3\,(\sin(t+\psi))^2\,(\cos(t+\psi))^2\right]\,dt \\ &= \epsilon^4\,\delta\,(-\,3\,\pi/2 + 3\pi/4) = -3\,\pi\,V_s^2\,\delta < 0.\end{aligned} \tag{11}$$

It is seen that, for small ϵ, the negative term indeed "wins" the "competition" mentioned previously, providing, in total, a stabilizing effect. However, it is also seen that, as V_s gets small, the fractional reduction of V_s with each successive orbit around the origin, $\Delta V_s/V_s \approx -3\,\pi\, V_s\,\delta$, reduces to zero (that is, the RHS is itself proportional to V_s), thus indicating only asymptotic convergence. We thus abandon the constant δ case, in favor of that considered in the next section.

3 Case with $\delta(t) = C/V_s(t)$

Given the observation in §2 that, using the control rule (4) with constant δ, the variations of the control input diminish to zero faster than the state oscillations themselves do, thus providing only asymptotic convergence [see (11) and Figure 1], we consider now the scaling of $\delta(t)$ with the inverse of $V_s(t)$, taking $\delta = C/V_s$, in order to maintain the amplitude of the oscillations of $\ell(t)$ even as the amplitude of the oscillations of $\phi(t)$ are diminished.

The analysis proceeds essentially as before, with a few more terms to deal with. By (4) with (5b), it now follows [cf. (6)] that

$$\dot\ell = (C/V_s)\,(\dot\phi^2 + \phi\,\ddot\phi) + \phi\,\dot\phi\,\dot\delta = (C/V_s)\,(\dot\phi^2 + \phi\,\ddot\phi) - (C/V_s^2)\,\phi\,\dot\phi^2\,(\ddot\phi + \sin\phi). \tag{12}$$

Given (1), (4), (5b), and (12), it thus now follows [cf. (7)] that

$$\begin{aligned}
[1 + (C/V_s)\,\phi\,\dot\phi]\,\ddot\phi + \sin\phi &= -2\,\dot\phi\,[(C/V_s)\,(\dot\phi^2 + \phi\,\ddot\phi) - (C/V_s^2)\,\phi\,\dot\phi^2\,(\ddot\phi + \sin\phi)],\\
[1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2]\,\ddot\phi + \sin\phi &= -(2\,C\,\dot\phi^3/V_s)\,[1 - \phi\,(\sin\phi)/V_s],\\
\ddot\phi + \frac{\sin\phi}{1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2} &= \frac{-(2\,C\,\dot\phi^3/V_s)\,[1 - \phi\,(\sin\phi)/V_s]}{1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2},\\
\ddot\phi + \sin\phi &= \frac{-(2\,C\,\dot\phi^3/V_s)\,[1 - \phi\,(\sin\phi)/V_s] + [3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2]\,\sin\phi}{1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2}.
\end{aligned} \tag{13}$$

We must again ensure that the denominator on the RHS of (13) remains positive, and thus that $\ddot\phi$ remains bounded. To accomplish this, we define $r = \dot\phi/\phi$ and, noting that $V_s/\phi^2 = (1 + r^2)/2$, write the denominator of the RHS of (13) as

$$1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2 \triangleq 1 + C f \quad\Rightarrow\quad f(r) = r\,[6/(1 + r^2) - 8\,r^2/(1 + r^2)^2].$$

Note that $f(r) \to 0$ as $r \to \pm\infty$. The global minimum value of $f(r)$, over $-\infty < r < \infty$, is easy to calculate:

$$f_{\min} \triangleq \min_r f(r) = -\big[207/128 + 33\sqrt{33}/128\big]^{1/2}/8 \approx -1.7602 \quad\text{at}\quad r = \bar r = -\big[6 - \sqrt{33}\big]^{1/2} \approx -0.50541.$$

We therefore ensure that $\ddot\phi$ remains bounded by taking $C < 1/|f_{\min}|$; that is, by assuming that

$$0 < C < 1/|f_{\min}| = C_{\max} = -[(69 - 11\sqrt{33})/2]^{1/2}/3 \approx 0.56813. \tag{A2}$$

This upper bound on the maximum allowable C is tight; for $C = C_{\max}$, when $\dot\phi/\phi = \bar r$, the denominator of (13) goes to zero, and $\ddot\phi$ diverges. For $C < C_{\max}$, no combinations of ϕ and $\dot\phi$ drive the denominator to zero. In practice, C should be kept significantly smaller than $C_{\max}$ in order to ensure a well-behaved controlled system, accurately governed by the model given in (1) with appropriately small accelerations $\ddot\phi$.

Again, whether the various RHS terms of (13) result, in total, in stabilization or destabilization of this oscillator is the central question of interest addressed below. Noting the definition of V_s in (5b), we now [cf. (8)] have

$$\begin{aligned}
\frac{dV_s}{dt} &= \dot\phi\,(\ddot\phi + \sin\phi) = \dot\phi\left(\frac{-(2\,C\,\dot\phi^3/V_s)\,[1 - \phi\,(\sin\phi)/V_s] + [3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2]\,\sin\phi}{1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2}\right)\\
&= \frac{-(2\,C\,\dot\phi^4/V_s)\,[1 - \phi\,(\sin\phi)/V_s] + [3\,C\,\phi\,\dot\phi^2/V_s - 2\,C\,\phi\,\dot\phi^4/V_s^2]\,\sin\phi}{1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2}.
\end{aligned} \tag{14}$$

It is seen in the (exact) expression above that, if $C = 0$, then $dV_s/dt = 0$, and thus V_s is a constant, set by the initial conditions on ϕ and $\dot\phi$. It follows from (14) [cf. (9)] that

$$\frac{dV_s}{dt} \le \frac{-2\,C\,\dot\phi^4/V_s + 2\,C\,\phi^2\,\dot\phi^4/V_s^2 + 3\,C\,\phi^2\,\dot\phi^2/V_s - 2\,C\,\phi^2\,\dot\phi^4/V_s^2}{1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2} = \frac{(C\,\dot\phi^2/V_s)(-2\,\dot\phi^2 + 3\,\phi^2)}{1 + 3\,C\,\phi\,\dot\phi/V_s - 2\,C\,\phi\,\dot\phi^3/V_s^2}. \tag{15}$$

Again, for small ϕ (and thus $\sin\phi \approx \phi$), the bound given in (15) is tight. It is seen that, assuming (A2), the time derivative of V_s along system trajectories, considered as a candidate Lyapunov function, is again bounded above by a competition of two terms, one negative semidefinite (stabilizing) and one positive semidefinite (destabilizing). Whenever $\phi^2 > (2/3)\dot{\phi}^2$, the magnitude of the destabilizing term becomes greater than that of the stabilizing term, and thus we can again not establish that $dV_s/dt \leq 0$ everywhere in any finite region surrounding the origin using this candidate Lyapunov function.

3.1 Analysis for small C

If we now also assume that both C and V_s are small, the terms added to unity in the denominator of (15) are negligible. In this case, V_s is nearly constant, and system trajectories are nearly circular. Integrating the RHS of (15) around 2π radians of the circular approximate trajectory $\{\varphi, \dot{\varphi}\}$ defined in (10) as before now gives

$$\begin{aligned} \Delta V_s = \int_0^{2\pi} (C/V_s)[-2\,\dot{\varphi}^4 + 3\,\varphi^2\,\dot{\varphi}^2]\,d\tau &= \epsilon^4\,(C/V_s)\int_0^{2\pi}\left[-2\,(\cos(t+\psi))^4 + 3\,(\sin(t+\psi))^2\,(\cos(t+\psi))^2\right]\,dt \\ &= 4\,V_s\,C\,(-\,3\,\pi/2 + 3\pi/4) = -3\pi\,V_s\,C < 0. \end{aligned} \tag{16}$$

As V_s gets small, the fractional reduction of V_s with each orbit around the origin, $\Delta V_s/V_s = -3\pi\,C$, now remains constant [cf. (11)], thus establishing exponential convergence, for small C and small V_s, in the discrete-time sense of [6] mentioned previously.

3.2 Analysis for finite C

As mentioned previously, for larger values of C, closed-form solutions of system trajectories, which spiral in to the origin [albeit, with non-monotonically decreasing $V_s(t)$, as illustrated in Figure 1c] are apparently not available. Note in particular [in Figure 1b] that, for small $\epsilon = \sqrt{2\,V_s}$, these solutions are *not* simply decaying circular orbits; this is because the (complicated) four terms in the numerator of the RHS of (13) are now also $O(\epsilon)$, and the three terms in the denominator of the RHS of (13) are all $O(\epsilon^0)$.

However, the full nonlinear EOM in this case, (13), is again easily integrated numerically using a standard RK4 approach with small Δt to determine $\phi(t)$. We perform such simulations for small initial conditions $\{\phi, \dot{\phi}\}_{t=0}$, and then calculate V_s both before and after a full (2π radians) revolution of the system trajectory about the origin in phase space. [This may be done either directly, via the definition of V_s in (5b), or indirectly, via integration of the RHS of (14) along system trajectories; both approaches converge, for small Δt, to the same answer.] We may represent the convergence rates in these simulations [cf. (16)] as

$$\Delta V_s/V_s = -\,p\,C = -f < 0. \tag{17}$$

Figure 2 plots the results of such tests for C ranging from 0.001 to 0.5. Note that $p \to 3\pi$ as $C \to 0$, as determined analytically in §3.1. For $C = 0.5$, the calculated value of f is 0.993.

As also illustrated in Figure 2c [and, quite convenient from the perspective of controller design], it follows from

$$\ell = 1 + \delta\,\phi\,\dot{\phi}, \quad \delta = C/V_s, \quad V_s = \dot{\phi}^2/2 + 1 - \cos\,\phi,$$

the fact that $V_s \approx (\dot{\phi}^2 + \phi^2)/2$ for small V_s, and the relation $|\phi\,\dot{\phi}| = \max_t\,(\phi^2 + \dot{\phi}^2)/2$ due to Cauchy Schwarz, that, for small V_s, for each oscillation of the controlled system,

$$C = \max_t\,|\ell(t) - 1|. \tag{18}$$

Note in these plots that the fractional reduction of V_s per complete revolution around the origin in phase space, $f = -\Delta V_s/V_s$ (in a sense, the "discrete-time rate of exponential convergence" in this system), increases for increasing C, approaching $f = 1.0$ (that is, driving V_s to zero in a single swing) as $C \to C_{\max}$. Recall also that C must be kept smaller than $C_{\max}$ [see (A2)] in order to keep $\ddot{\phi}(t)$ from diverging; practical constraints might limit control solutions even further. For example, in practical application, keeping C [see (18)] to somewhere around, say, 1% to 10%, might be preferred. As shown in Figure 2, this corresponds to fractional reductions of V_s per revolution around the origin in phase space of 9% to 61%, which is still quite substantial. For example, a fractional reduction rate $\Delta V_s/V_s$ of 61% corresponds to reducing V_s by an order of magnitude every $\ln(0.1)/\ln(1-0.61) = 2.445$ revolutions around the origin in phase space, which for small V_s again takes $T_0 \approx 2\pi$ nondimensional time units. For reference, the black line in Figure 1c indicates a slope corresponding to a reduction of $V_s(t)$ by an order of magnitude every $\Delta t = 2.445 \cdot 2\pi$ nondimensional time units, which matches well the sustained exponential rate of reduction of V_s seen in simulation for $C = 0.1$.

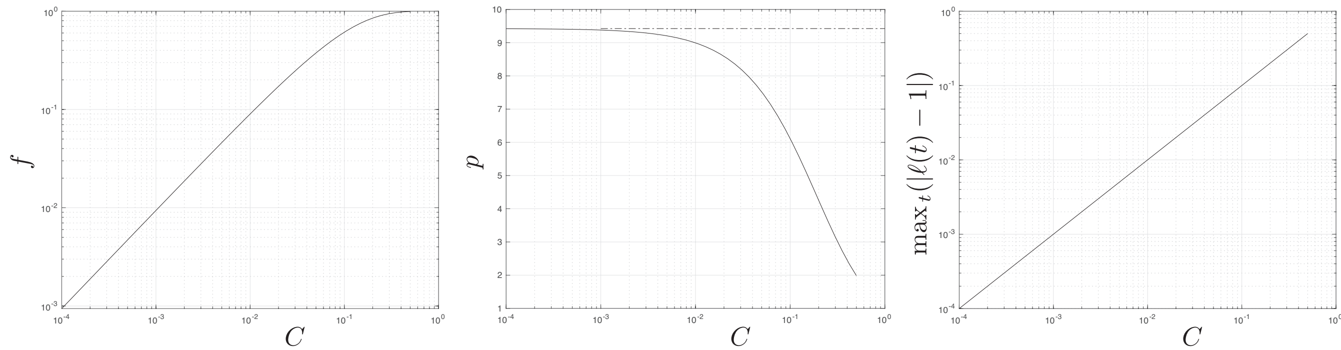

Figure 2: Exponential convergence behavior of present control approach for small V_s and finite C, with $0.0001 \leq C \leq 0.5$. (a) The fractional reduction of V_s per revolution around the origin in phase space, $f = -\Delta V_s / V_s$, versus C. (b) The coefficient $p = f/C$ in (17) versus C [note that, for diminishing C, $p \to 3\pi$, as indicated by the dot-dashed curve, and as determined analytically in (16)]. (c) The corresponding maximum magnitude of the deviations of the pendulum length, $\max_t(|\ell(t) - 1|)$, versus C [note that, conveniently, $C = \max_t(|\ell(t) - 1|)$].

3.3 The "curious caveat"

As for the curious caveat, note that the above analysis only establishes that V_s [see (5b)] converges exponentially (in a discrete-time sense); the control inputs [that is, the fluctuations of $\ell(t)$] actually remain $O(1)$ as $\{\phi, \dot{\phi}\}$ converge to zero as long as the control is applied. This must in fact be the case if exponential convergence is sought, as the control term $\dot{\ell}$ multiplies $\dot{\phi}$ as it enters the evolution equation (1); to remain effective even as the oscillations of $\{\phi, \dot{\phi}\}$ get small, it is necessary that the fluctuations of $\ell(t)$ remain $O(1)$.

4 Conclusions

Combining the varpend oscillator (1) with the control rule (4) taking (5a) leads to an EOM that may be written as (7); assumption (A1) is necessary and sufficient to ensure that $\ddot{\phi}$ is bounded. In this case, dV_s/dt is given (exactly) by (8), and a sharp upper bound on dV_s/dt is given by (9), which exhibits a competition between two terms, one negative semidefinite, and one positive semidefinite. Though the resulting $V_s(t)$ indeed decreases over time (albeit, non-monotonically), convergence in this case is found to be only asymptotic (see Figures 1a and c). This is consistent with the fact that, after each full revolution of the system trajectory about the origin in phase space, the fractional decrease of V_s in this case is found to approach $\Delta V_s/V_s \approx -3\,\pi\,V_s\,\delta$ [as determined analytically in (11)], which diminishes as V_s is reduced.

Combining (1) with (4) taking (5b), on the other hand, leads to an EOM that may be written as (13); assumption (A2) is necessary and sufficient to ensure that $\ddot{\phi}$ is bounded. In this case, dV_s/dt is given (exactly) by (14), and a sharp upper bound on dV_s/dt is given by (15), which again exhibits a competition between two sign-semidefinite terms. The resulting $V_s(t)$ in this case decreases over time exponentially, albeit (again) non-monotonically (see Figures 1b and c). After each full revolution of the system trajectory about the origin in phase space, the fractional decrease of V_s is found in this case to approach $\Delta V_s/V_s \approx -p\,C$ [see (11)], with $3\pi > p > 2$ for $0 < C < 0.5$, with (as determined analytically) $p \to 3\pi$ as $C \to 0$. The fluctuations of $\ell(t)$ actually remain $O(1)$ in this case as $\{\phi, \dot{\phi}\}$ converge towards zero. In application, once the oscillations of $\{\phi, \dot{\phi}\}$ diminish (exponentially) to the desired value, one may simply (smoothly) reduce C to zero, and thus [by (4) and (5b)] $\ell(t)$ will return (smoothly) to 1 [that is, $L(t)$ will return (smoothly) to its target value L_0].

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