Constructing Control Lyapunov-Value Functions Using Hamilton-Jacobi Reachability Analysis

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***Abstract*—In this letter, we seek to build connections between control Lyapunov functions (CLFs) and Hamilton-Jacobi (HJ) reachability analysis. CLFs have been used extensively in the control community for synthesizing stabilizing feedback controllers. However, there is no systematic way to construct CLFs for general nonlinear systems and the problem can become more complex with input constraints. HJ reachability is a formal method that can be used to guarantee safety or reachability for general nonlinear systems with input constraints. The main drawback is the well-known "curse of dimensionality." In this letter we modify HJ reachability to construct what we call a control Lyapunov-Value Function (CLVF) which can be used to find and stabilize to the smallest control invariant set ($\mathcal{I}_m^\infty$) around a point of interest. We prove that the CLVF is the viscosity solution to a modified HJ variational inequality (VI), and can be computed numerically, during which the input constraints and exponential decay rate γ are incorporated. This process identifies the region of exponential stability to $\mathcal{I}_m^\infty$ given the desired input bounds and γ. Finally, a feasibility-guaranteed quadratic program (QP) is proposed for online implementation.**

***Index Terms*—Optimal control, nonlinear systems, asymptotic stability, quadratic programming.**

I. Introduction

AUTONOMOUS systems performing tasks in the real world should be both *live* (able to complete tasks) and *safe*. Control Lyapunov functions (CLFs) are a popular method to ensure liveness by stabilizing trajectories of a system to an equilibrium point [1], [2], [3]. Control barrier functions (CBFs) on the other hand are used to guarantee safety by maintaining trajectories of a system within a safe control invariant set [4], [5], [6]. Unfortunately, finding CLFs and CBFs is difficult: there lacks universal construction methods that work for general nonlinear systems. Hand-designed or application-specific CLFs and CBFs can be used [7], [8], [9], [10], [11]. However, these hand-crafted functions can be conservative in many cases, and may be invalid when faced with input bounds.

Liveness and safety can also be achieved by formal methods such as Hamilton-Jacobi (HJ) reachability analysis [12], [13], [14], [15], [16]. This method computes a value function whose level sets provide information about safety (liveness) over space and time, and whose gradients provide the safety (liveness) controller. This value function can be computed numerically using dynamic programming, handles general nonlinear systems, and can accommodate input and disturbance bounds. Undermining these appealing benefits is the "curse of dimensionality." Ongoing research has improved computational efficiency [17], [18], [19], but performing dynamic programming in high dimensions (6D or more) remains challenging. Additionally, standard HJ reachability analysis cannot stabilize a system to a goal.

Recent work has shown that CBF-like functions can be constructed by modifying HJ reachability analysis [20]. In this letter we seek to extend this work to constructing control Lyapunov-value functions (CLVFs), which have similar stabilizing properties to CLFs. This extension is nontrivial: HJ reachability applied to liveness traditionally seeks to find the minimum time to reach a goal (and may be forced to exit afterwards), whereas a CLVF seeks to stabilize to a goal. Additionally, there are many systems for which a valid CLF does not exist due to no stabilizable equilibrium points. For such systems, the CLVF can find and stabilize to the smallest control invariant set around a point of interest.However, the CLVF also suffers the "curse of dimensionality."

In this letter, the main contributions are:

1) We define the CLVF and establish the theoretical foundation of how to compute this function.
2) We establish the relation between CLVFs and the exponential stabilizability of nonlinear systems. We provide a numerical estimation of the region of exponential stabilizability (ROES), and demonstrate the effect of the exponential rate parameter γ on the ROES.
3) For systems with no stabilizable equilibrium points, we show that the CLVF stabilizes the system to its smallest control invariant set (if one exists).
4) We provide a Quadratic Program (QP)-based controller, and show that both feasibility and the exponential decay rate are guaranteed.

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II. Background

A. Problem Formulation

Consider the general nonlinear time-invariant system

$$\dot{x}(s) = f(x(s), u(s)), \quad s \in [t, 0], \quad x(t) = x, \tag{1}$$

where $t < 0$ is the initial time, $x \in \mathbb{R}^n$ is the initial state. We follow the convention in HJ reachability literature and set 0 as the final time. The control input u is drawn from a compact set $\in \mathcal{U} \subset \mathbb{R}^m$, and the control function $u(\cdot)$ is assumed to be drawn from the set of measurable functions $\mathbb{U} : [t, 0] \mapsto \mathcal{U}$. Assuming that the dynamics $f : \mathbb{R}^n \times \mathcal{U} \mapsto \mathbb{R}^n$ is Lipschitz continuous in (x, u) and continuous in the s, there exists a unique solution $\varphi(s) = \varphi(s; x, t, u(\cdot)) : [t, 0] \mapsto \mathbb{R}^n$ of the system (1) given initial state x and control function $u(\cdot)$.

Systems with input constraints may admit several control invariant sets. For systems with a stabilizable equilibrium point, the equilibrium point itself is the smallest control invariant set. Systems with no equilibrium point may still have control invariant sets. In both cases, we denoted the smallest control invariant set as $\mathcal{I}_m^\infty$, and assume it contains the origin. In this letter, we seek to exponentially stabilize the system to its smallest control invariant set.

B. Optimal Control and HJ Reachability

In this letter we introduce the safety (rather than liveness) formulation of HJ reachability. This is because in this letter we construct the *liveness* problem of stabilizing to the smallest control invariant set $\mathcal{I}_m^\infty$ as a *safety* problem, wherein the system seeks to avoid all regions of the state space that are not $\mathcal{I}_m^\infty$.

To compute the value function for HJ reachability we define a Lipschitz continuous cost function $\ell : \mathbb{R}^n \mapsto \mathbb{R}$ whose super-zero level set is the failure set $\mathcal{F} = \{x : \ell(x) \geq 0\}$. The finite-time horizon cost function captures whether a given trajectory enters $\mathcal{F}$ at any point in the time horizon:

$$J(t, x, u) = \max_{s \in [t,0]} \ell(\varphi(s; x, u, t)). \tag{2}$$

The value function is this cost given optimal control:

$$V(x, t) = \min_{u \in \mathbb{U}_{[t,0]}} J(t, x, u) = \min_{u \in \mathbb{U}_{[t,0]}} \max_{s \in [t,0]} \ell(\varphi(s)). \tag{3}$$

This value function is the unique Lipschitz continuous viscosity solution to the following Hamilton-Jacobi-Bellman variational inequality (HJB-VI) [21]:

$$\max\left\{\ell(x) - V(x, t), D_t V(x, t) + \min_{u \in \mathbb{U}_{[t,0]}} D_x V(x, t) \cdot f(x, u)\right\} = 0. \tag{4}$$

Therefore the value function can be computed using dynamic programming by applying this HJB-VI recursively over time. The infinite-time horizon value function is defined by taking the limit of $V(x, t)$ as $t \to -\infty$ [22],

$$V^\infty(x) = \lim_{t \to -\infty} V(x, t). \tag{5}$$

For the time-varying value function, trajectories that start in the super-zero level set $V(x, t) \geq 0$ will enter $\mathcal{F}$ for some time $s \in [t, 0]$. The sub-zero level set of $V(x, t)$ is therefore safe for the time horizon. This can be extended to say that each sub-α level set $\mathcal{V}_\alpha = \{x : V(x, t) \leq \alpha\}$ is safe with respect to the set defined by $\mathcal{F}_\alpha = \{x : \ell(x) \leq \alpha\}$. In the infinite-time setting, this means every sub-α level set of $V^\infty(x)$ is control invariant and can maintain trajectories within a particular level set boundary. Further more, if $\ell(x)$ is chosen such that $\ell(0) = 0$ and $\forall x \neq 0$, $\ell(x) > 0$, the sub level set corresponding to the minimum value of $V^\infty(x)$ is the smallest control invariant set $\mathcal{I}_m^\infty$ of the system.

Definition 1: A set $\mathcal{I}^\infty$ is infinite-time control invariant if $\forall x \in \mathcal{I}^\infty$, there exists some control $u(\cdot)$ such that as $t \to -\infty$, $\forall s \in [t, 0]$, $\varphi(s; x, u, t) \in \mathcal{I}^\infty$.

Definition 2: The ROES of a control invariant set $\mathcal{I}^\infty$ is defined as

$$\mathcal{D}_{\text{ROES}} := \Big\{x \in \mathbb{R}^n | \exists u \in \mathbb{U}, \gamma > 0 \text{ s.t. } ||\varphi(s; x, u, t)|| \to \mathcal{I}^\infty \text{ with exponential rate } \gamma \text{ as } t \to -\infty\Big\}.$$

However, since the set is only control invariant, there is no guarantee that the system can be stabilized to lower level sets or the origin.

C. Control Lyapunov Functions

CLFs are a common tool for ensuring and enforcing the stability of a system. A continuous function $V_{\text{clf}} : \mathbb{R}^n \to \mathbb{R}$ is a local CLF for the equilibrium point $\bar{x}$ if, in a neighborhood $\mathcal{O}$ of $\bar{x}$, the following holds: (a) V_{clf} is proper at $\bar{x}$, (b) V_{clf} is positive definite on $\mathcal{O}$ and continuously differentiable on $\mathcal{O}$, and (c) for each $x \in \mathcal{O}$, there exists some $u \in \mathcal{U}$ such that $\dot{V}_{\text{clf}}(x) = \frac{dV_{\text{clf}}}{dx} \cdot f(x, u) < 0$.

The existence of a CLF implies the system is asymptotically stabilizable, and the gradient of the CLF can be used to generate a stabilizing controller. If in addition $\dot{V} \leq \gamma V$ for some $\gamma > 0$, exponential stabilizability is guaranteed.

For a local CLF, each sub-level set contained in $\mathcal{O}$ is not only control invariant, but also attractive. Informally speaking, control invariance only guarantees not leaving the control invariant set, whereas attractive guarantees shrinking to a smaller set. This appealing property differs from the sub-level set of the infinite-time value function defined in HJ reachability analysis (5). In this letter we seek to modify (5) to obtain this property.

Note that many nonlinear systems fail to have a continuously differentiable CLF, though there has been work to generalize CLFs to only be continuous based on generalized gradients [23], [24].

D. Control Barrier-Value Functions

Control barrier functions are similar to CLFs but are generated for safety problems [4]. Recent work [20] introduced a method of modifying HJ reachability value functions to have similar properties to CBFs. The resulting control barrier value function (CBVF) is the unique Lipschitz continuous viscosity solution to a specific variational inequality called CBVF-VI, and has the property $\dot{B}_\gamma \geq -\gamma B_\gamma$. The CBVF gives the largest control invariant set, and the optimal control is less conservative compared to classic HJ reachability [20].

We emphasize some key differences between the CBVF and our proposed CLVF: 1) The CBVF focuses on staying within

Algorithm 1: Obtaining the CLVF for General Nonlinear Systems (Offline)

1. **Input**: System model, input bounds, desired exponential rate $\gamma > 0$.
2. **Output**: CLVF $V_\gamma^\infty(x)$.
3. Find the smallest control invariant set: take $\ell_1(x) = ||x||_2$ and $\gamma_1 = 0$, use (8) to get $V^\infty(x)$
4. Compute the CLVF: $a \leftarrow \min_x V^\infty(x)$, take $\ell_2(x) = ||x||_2 - a$, use (8) to get $V_\gamma^\infty(x)$.

the safe set in finite time, while we focus on stabilizing to the goal over an infinite-time horizon. 2) The CBVF is defined on $\mathbb{R}^n$, as is our time-varying CLVF (TV-CLVF), but the infinite-time CLVF is not guaranteed to be defined on $\mathbb{R}^n$. 3) We prove the CLVF is the viscosity solution to the CLVF-VI in the infinite-time horizon, whereas the CBVF is only proved for the time-varying case. 4) We propose the CLVF-QP which is guaranteed to be feasible for all time, while the CBVF-QP can only guarantee feasibility for a finite-time.

III. Control Lyapunov-Value Functions

In this section we introduce the CLVF and its properties.

A. Definition of CLVFs

Definition 3: A Time-Varying Control Lyapunov-Value Function (TV-CLVF) $V_\gamma(x,t) : \mathbb{R}^n \times \mathbb{R}_- \mapsto \mathbb{R}_+$, is defined as

$$V_\gamma(x,t) = \min_{u\in\mathbb{U}_{[t,0]}} \max_{s\in[t,0]} e^{\gamma(s-t)}\ell(\varphi(s;x,t,u)), \tag{6}$$

where $J_\gamma(t,x,u) = \max_{s\in[t,0]} e^{\gamma(s-t)}\ell(\varphi(s;x,t,u))$ is the finite-time cost, γ is a user specified non-negative parameter which represents the desired decay rate, $\ell(x) = ||x||_2 - a$, and a is a constant determined by Algorithm 1. The control seeks to decrease the system along $\ell(x)$ towards the origin. Note that the specification of $\ell(x)$ does not restrict the TV-CLVF to be in quadratic form.

The standard HJ reachability value function (3) is a special case of (6) where $\gamma = 0$. We will show how the inclusion of this exponential term provides the region for which the system can be stabilized at a rate of γ.

Note that since $\forall s \in [t,0]$ and $\gamma \geq 0$, $e^{\gamma(s-t)} \geq 1$, we have $V_\gamma(x,t) \geq V_0(x,t)$. Therefore each level set of $V_\gamma(x,t)$ is a subset of the level set of $V_0(x,t)$.

We extend the definition of the TV-CLVF to infinite time:

Definition 4: Given a compact set $\mathcal{D}_\gamma$, the function $V_\gamma^\infty : \mathcal{D}_\gamma \subset \mathbb{R}^n \mapsto \mathbb{R}_+$ is a CLVF if the following limit exists:

$$V_\gamma^\infty(x) = \lim_{t\to-\infty} V_\gamma(x,t). \tag{7}$$

The existence of the limit in (7) implies that the CLVF is unique given a specific system and input bounds.

Remark 1: Note that the existence of the limit to negative infinity in (7) is equivalent to $\lim_{n\to\infty} V_\gamma(x,t_n) = V_\gamma^\infty(x)$ for every sequence $\{t_n\}$ s.t. $t_n \neq -\infty$ and $\lim_{n\to\infty} t_n = -\infty$. In other words, if we evaluate $V_\gamma(x,t)$ at each t_n, and get a sequence of functions $\{U_n\}$ such that $U_n(x) = V_\gamma^\infty(x,t_n)$, then $\{U_n\}$ converges to V_γ^∞ pointwise in $\mathcal{D}_\gamma$.

If in addition, the sequence $\{t_n\}$ is monotonically decreasing, by Dini's Theorem, the convergence of $\{U_n\}$ to V_γ^∞ is uniform in $\mathcal{D}_\gamma$.

B. Dynamic Programming Principle & Viscosity Solution

We next establish the theoretical foundation for computing the CLVF in two steps. First, we show that Bellman's optimality principle can be used to derive the dynamic programming principle for the CLVF. Second, we show that the CLVF is the viscosity solution to the CLVF-VI.

Theorem 1: For all $t_1 > 0$, $\forall u \in \mathbb{U}$ and $z = \varphi(t_1; x, 0, u)$, the following is satisfied

$$V_\gamma^\infty(x) = \min_{u\in\mathbb{U}} \max\left\{ \max_{s\in[0,t_1]} e^{\gamma t_1}\ell(\varphi(s;x,0,u)), e^{\gamma t_1} V_\gamma^\infty(z)\right\}. \tag{8}$$

The proof is analogous to the proof in [20], but with certain differences, as we proved for the infinite-time horizon. We provide a detailed proof in [25].

Theorem 2: The CLVF is the unique viscosity solution to the following CLVF-VI,

$$\max\left\{\ell(x) - V_\gamma^\infty(x), \min_{u\in\mathbb{U}} D_x V_\gamma^\infty \cdot f(x,u) + \gamma V_\gamma^\infty(x)\right\} = 0. \tag{9}$$

The proof is analogous to the proofs in [13], [14], [15]. For details refer to [25].

Note the differences between (9) and (4) are: first, (9) is time-invariant and second, the $\gamma V_\gamma^\infty(x)$ in (9).

C. Algorithm for Computing CLVF

Here we present an Algorithm to compute the CLVF. Specifically, we emphasize how the constant a in $\ell(x)$ is determined. Remark 2 and Proposition 1 are vital.

Remark 2: In classic HJ reachability, an interesting observation is that

$$\begin{aligned} V(x,t) + c &= \min_{u\in\mathbb{U}_{[t,0]}} \max_{s\in[t,0]} \left\{\ell(\varphi(s))\right\} + c \\ &= \min_{u\in\mathbb{U}_{[t,0]}} \max_{s\in[t,0]} \left\{\ell(\varphi(s)) + c\right\}. \end{aligned}$$

This means for a given system, at any pair (x,t), adding a constant value to the terminal cost $\ell(x)$, the resulting value function will also be added with the same constant. However, this is generally not true for the CLVF.

Proposition 1: For a given system and fixed terminal cost $\ell(x)$, CLVFs with different γ's have the same zero-level set.

The offline computation of CLVFs is shown in Algorithm 1. Line 3 computes a classic HJ infinite time value function with $\gamma = 0$. If this value function exists, the level set of its minimum value is $\mathcal{I}_m^\infty$ of this system given input bounds. Line 4 re-initializes the computation to compute the CLVF. By Remark 2, if in Line 4 we again set $\gamma = 0$, the zero level set of the resulting function will be $\mathcal{I}_m^\infty$. By Proposition 1, the zero level set of CLVF with a positive γ will be $\mathcal{I}_m^\infty$. This means the CLVF is positive outside $\mathcal{I}_m^\infty$, and zero inside.

To compute the limits in practice, note that the value of a state will either converge to a finite number, or diverge to infinity. For a given state, if the **change of the value** between two intermediate time steps is smaller than the convergence threshold, it is characterized as converged; if the **value** is greater than the divergence threshold it is characterized as diverged. The computation is ran until all the states have either converged or diverged. Using this criteria, some theoretically converging states may be misclassified as diverging, resulting in an under-approximation of the ROES.

D. Infinite-Time CLVF and Exponential Stabilizability

Here we establish the relationship between the exponential stabilizability to $\mathcal{I}_m^\infty$ and the convergence of the CLVF. We show that the CLVF will converge locally (globally) if and only if $\mathcal{I}_m^\infty$ is locally (globally) exponentially stabilizable. The following proposition will help us establish the result.

Proposition 2: At any point (differentiable or non-differentiable) in the domain $\mathcal{D}_\gamma$ of the CLVF, there exists some control $u \in \mathcal{U}$ such that

$$\dot{V}_\gamma^\infty \leq -\gamma V_\gamma^\infty. \tag{10}$$

This can be proved using [16, Th. 2.3]. We provide a detailed proof in [25] and omit it here.

Theorem 3: The system can be exponentially stabilized to its smallest control invariant set $\mathcal{I}_m^\infty$ from $\mathcal{D}_\gamma \setminus \mathcal{I}_m^\infty$ (or $\mathbb{R}^n \setminus \mathcal{I}_m^\infty$), if and only if the CLVF exists in $\mathcal{D}_\gamma$ (or $\mathbb{R}^n$).

Proof: ($\leftarrow$) Assume the limit in (7) exists in $\mathcal{D}_\gamma$. For any initial state $x_0 \in \mathcal{D}_\gamma \setminus \mathcal{I}_m^\infty$, consider the optimal trajectory $\varphi(s) = \varphi(s; x, u^*, t)\ \forall s \in [t, 0]$. From Proposition 2:

$$D_x V_\gamma^\infty(x) \cdot f(x, u^*) = \dot{V}_\gamma^\infty \leq -\gamma V_\gamma^\infty. \tag{11}$$

Using the comparison principle, we have $\forall s \in [t, 0]$,

$$V_\gamma^\infty\big(\varphi(s)\big) \geq e^{-\gamma(s-0)} V_\gamma^\infty\big(\varphi(0)\big). \tag{12}$$

Applying the definition of TV-CLVF,

$$V_\gamma\big(\varphi(0), 0\big) = \ell\big(\varphi(0)\big) = ||\varphi(0)|| - a.$$

Combining the fact that $V_\gamma(x, t) \leq V_\gamma^\infty(x)$, we have:

$$||\varphi(0)|| = V_\gamma\big(\varphi(0), 0\big) + a \leq V_\gamma^\infty\big(\varphi(0)\big) + a.$$

Applying (12) gives us

$$\begin{aligned} ||\varphi(0)|| &\leq V_\gamma^\infty\big(\varphi(0)\big) + a \leq e^{\gamma s} V_\gamma^\infty\big(\varphi(s)\big) + a \\ &= \frac{e^{-\gamma(0-s)} V_\gamma^\infty\big(\varphi(s)\big)}{||\varphi(s)||} ||\varphi(s)|| + a. \end{aligned}$$

This inequality holds for all $s \in [t, 0]$, so it holds for $s = t$:

$$\begin{aligned} ||\varphi(0)|| &\leq \frac{e^{-\gamma(0-t)} V_\gamma^\infty\big(\varphi(t)\big)}{||\varphi(t)||} ||\varphi(t)|| + a \\ &= e^{-\gamma(0-t)} k ||\varphi(t)|| + a \end{aligned}$$

where $k = \frac{V_\gamma^\infty(\varphi(t))}{||\varphi(t)||}$. Since the limit in (7) exists, we know that $V_\gamma^\infty(\varphi(t)\ )$ is finite. Additionally, $||\varphi(t)||$ is the distance to the origin of the initial state, which is also finite. This inequality holds for all $\varphi(t) \in \mathcal{D}_\gamma \setminus \mathcal{I}_m^\infty$. Therefore, we conclude that $\forall \varphi(t) \in \mathcal{D}_\gamma \setminus \mathcal{I}_m^\infty$, there exist some constants k and γ such that

$$||\varphi(0)|| \leq e^{-\gamma(0-t)} k ||\varphi(t)|| + a.$$

In other words, the controlled system can be locally exponentially stabilized to $\mathcal{I}_m^\infty$. If the limit exists in $\mathbb{R}^n$, we conclude that the above result holds globally.

($\rightarrow$) Again, consider the optimal solution $\varphi(s) = \varphi(s; x, u^*, t)\ \forall s \in [t, 0]$. Assume exist some constants k and γ and a set $\mathcal{D}_\gamma$ such that $\forall s \in [t, 0]$ and $\forall \varphi(s) \in \mathcal{D}_\gamma$,

$$||\varphi(s)|| - a \leq k ||\varphi(t)|| e^{-\gamma(s-t)}. \tag{13}$$

Then, plugging (13) into (6),

$$\begin{aligned} V_\gamma\big(x, t\big) &= \max_{s \in [t,0]} e^{\gamma(s-t)} \Big(||\varphi(s)|| - a \Big) \\ &\leq \max_{s \in [t,0]} e^{\gamma(s-t)} (k ||\varphi(t)|| e^{-\gamma(s-t)}) = k ||\varphi(t)||. \end{aligned}$$

As $t \to -\infty$, $V_\gamma(\varphi, t)$ monotonically increases by definition, and is upper bounded by $k||\varphi(t)||$, we conclude that the infinite-time CLVF exists in $\mathcal{D}_\gamma$. If the above conditions hold globally, the infinite-time CLVF exists globally. ∎

Theorem 3 establishes the equivalence between the existence of CLVF and the exponential stabilizability of the systems. It implies that $\mathcal{D}_\gamma \setminus \mathcal{I}_m^\infty = \mathcal{D}_{ROES}$.

The main benefits of using CLVF are 1) each sub-level set of CLVF is not only control invariant, but also attractive, and the decay rate is exponential. 2) For systems with control invariant sets, trajectories within $\mathcal{D}_\gamma$ can be stabilized to $\mathcal{I}_m^\infty$. 3) The CLVF can be constructed numerically, removing the need for "guess and check" methods.

IV. Feasibility Guaranteed CLVF-QP

In this section, we specify one way of synthesizing the optimal control for control affine systems using QP. We emphasize that the proposed QP is guaranteed to be feasible $\forall x \in \mathcal{D}_\gamma$. A control affine system is of the form

$$\dot{x}(s) = f\big(x(s), u(s)\big) = g\big(x(s)\ \big) + h\big(x(s)\ \big)u(s), \tag{14}$$

where $g : \mathbb{R}^n \mapsto \mathbb{R}^n$, $h : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$. For such systems, (10) is equivalent to the following linear inequality in u:

$$D_x V_\gamma^\infty(x) \cdot g(x) + \min_{u \in \mathcal{U}} D_x V_\gamma^\infty(x)\ \ \cdot h(x) u \leq -\gamma V_\gamma^\infty(x).$$

Similar to the CLF-QP, we synthesize the optimal controller based on a pointwise min-norm optimization problem with one linear inequality constraint. If in addition, the input bound $\mathcal{U}$ is polytopic, the optimization becomes a QP.

Theorem 4 (Feasibility Guaranteed CLVF-QP): Given some reference control u_r, the optimal controller can be synthesized by the following CLVF-QP with guaranteed feasibility $\forall x \in \mathcal{D}_\gamma$.

$$\begin{aligned} &\min_{u \in \mathcal{U}} \quad (u - u_r)^T (u - u_r) \\ &\quad \text{s.t.} \quad D_x V_\gamma^\infty(x) \cdot g(x) + D_x V_\gamma^\infty(x) \cdot h(x) u \leq -\gamma V_\gamma^\infty(x) \end{aligned}$$

Proof: This is a direct result from Proposition 2. ∎

Remark 3: Solving this QP for every point $x \in \mathcal{D}_\gamma$, an optimal state feedback controller can be obtained, i.e., $u_{QP} = K(x)$. Though the QP is feasible, $K : \mathbb{D}_\gamma \to \mathcal{U}$ is not necessarily continuous. This comes from the non-smooth nature of the CLVF.

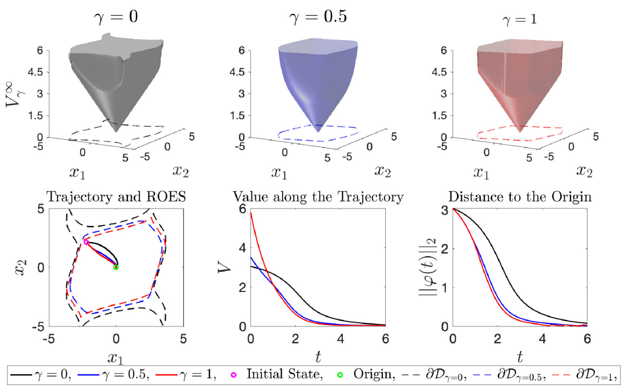

Fig. 1. An illustrative example of the CLVF approach. First row: given a system (15) with control bounds, for different desired exponentially stabilizing rates γ, the computed CLVFs and corresponding regions of exponential stabilizability (ROES) $\mathcal{D}_\gamma$'s are shown. It can be seen that the CLVFs are not smooth and not in quadratic form. When $\gamma = 0$, the ROES of CLVF is its largest control invariant set, and when $\gamma > 0$, the ROES contains all the states that can be stabilized to the origin with exponential rate γ. A larger γ results in smaller $\mathcal{D}_\gamma$. Second row: (left) trajectories using the CLVF-QP controller starting from the same initial condition, (middle) the values of the CLVFs decrease along these trajectories. A non-zero γ forces the value to decay; a larger γ results in a faster decrease. In this example, the dynamics are attractive, and drives the trajectory of $\gamma = 0$ to the origin.(Right) the distance to the origin of these trajectories. The same trend as the middle figure is observed.

V. Numerical Examples

The following examples illustrate the main benefits of our proposed method: feasibility guarantees, input bounds, and applicability to systems with no equilibrium point. We also study the effect γ on the ROES. All simulation results are based on MATLAB and tool boxes [26], [27]. The reference control for the QP controller is always chosen as $u_r = 0$.

A. Effect of γ

Consider the system given by

$$\begin{aligned} \dot{x}_1 &= -\sin(x_1) - 0.5\sin(x_1 - x_2), \\ \dot{x}_2 &= -0.5\sin(x_2) - 0.5\sin(x_2 - x_1) + u, \end{aligned} \tag{15}$$

where $u \in [-0.5, 0.5]$. It can be verified that $(x, u) = (0, 0)$ is an equilibrium point. We compute the CLVF for different γs and provide the estimate (under-approximation) of the corresponding ROESs. The computation times for constructing the CLVFs are on the order of $0.5 - 5$ minutes on a 201×201 grid. The CLVF-QP is used to calculate the optimal trajectories with initial state $x = [-2.2, 2.1]$. Results are shown in Fig. 1. Larger γ's lead to faster stabilization but a smaller ROES, allowing one to trade off aggressiveness of the controller with the region that can be stabilized.

B. Comparison With Back Stepping and Hand-Crafted CLF

Consider the following system

$$\dot{x}_1 = -\frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 - x_2, \quad \dot{x}_2 = u.$$

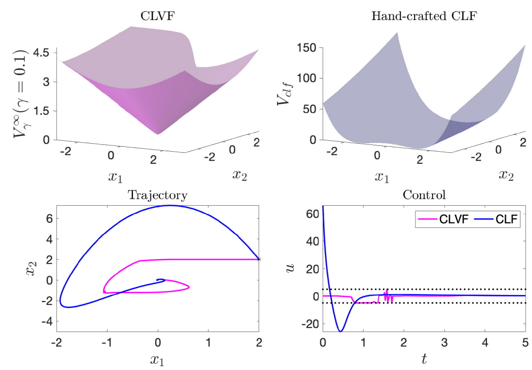

Fig. 2. Comparison of the CLVF (top-left) and the hand-crafted CLF (top-right). The input constraint is $u \in [-5, 5]$, with $\gamma = 1$. Bottom: comparison of trajectories (left) and control signal (right) using back stepping (red) and the CLVF-QP (blue). When using backstepping controller, the control is unbounded, and there is no guarantee of the convergence rate.

One can find a global CLF (if no input constraints) using back stepping: $V_{\text{clf}} = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \frac{3}{2}x_1^2)^2$, and the corresponding controller is: $u(x) = 3x_1(\frac{3}{2}x_1^2 + \frac{1}{2}x_1^3 + x_2) + x_1 - (x_2 + \frac{3}{2}x_1^2)$. Finding this CLF requires some expertise, and may not be valid under input constraints. However, a CLVF with input constraints can be directly computed as shown in Fig. 2. The computation time to construct this CLVF is 170.44 seconds on a 101×101 grid. Due to the non-smooth nature of the viscosity solution, the corresponding feedback law is not necessarily continuous.

C. Systems Without an Equilibrium Point

Consider the following dubins car example

$$\dot{x}_1 = v\cos(x_3), \quad \dot{x}_2 = v\sin(x_3), \quad \dot{x}_3 = u,$$

where $v = 1$ and $u \in [-\pi, \pi]$. This system cannot be stabilized to a point and therefore does not admit an equilibrium point, nor a CLF. However, our method can identify and stabilize to the smallest region that the system can stay within. The result is shown in Fig. 3, where for visualization the projection through the union of x_3 is shown. The computation time to construct this CLVF is 131.76 seconds on a $51 \times 51 \times 61$ grid.

VI. Conclusion

In this letter we proposed a method of finding a Lipschitz continuous CLVF which has properties similar to CLFs and can be computed numerically using dynamic programming. The proposed method can be applied to general nonlinear control systems. For control-affine systems, a feasibility guaranteed CLVF-QP is proposed to synthesize optimal controllers online.

The CLVF approach has two particularly useful properties. First, for systems with no equilibrium points, the proposed method can stabilize the system to the smallest control invariant set (if one exists) around a point of interest. Second, given a desired stabilizability rate γ and control bounds, this method can find the true ROESs. It should be noted that due to numerical issue, the ROESs shown in this letter are under

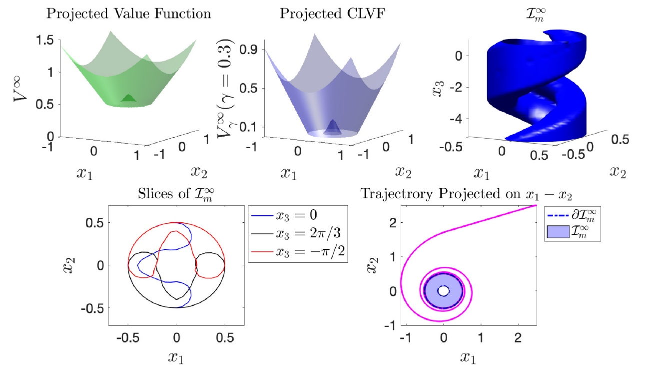

Fig. 3. These plots show how to use Algorithm 1 to find a positive semi-definite CLVF, and show the exponential decay rate of the value along the trajectory. Top-left using step 1 (Line 3 in Algorithm 1), the projected CLVF is shown with $\gamma = 0$. The minimum value is $V_{\min} = 0.51$. Top-middle: using step 2 (Line 4), the projected CLVF is shown with $\gamma = 0.3$ and $\ell(x) = ||x||_2 - 0.51$. In this example, we project the third dimension, and take it's minimum value, i.e., $V_{\text{proj}}(x_1, x_2) = \min_{x_3} V(x_1, x_2, x_3)$. Top-right: the smallest control invariant set $\mathcal{I}_m^\infty$ of the dubins car example. Bottow-right: three slices of $\mathcal{I}_m^\infty$ at different angles x_3. Stacking all the slices results in two concentric circles, as shown in bottow-right. Bottom-right: Using the CLVF-QP, the trajectory starting from $x_0 = [2.5, 2.5]$ converges to $\mathcal{I}_m^\infty$. The trajectory is approaching $\mathcal{I}_m^\infty$ in a stabilizing way.

approximation. One can change the parameter γ to trade off the rate under which the system can be stabilized with the region from which the system can be stabilized.

The main drawback to this method is the heavy computation requirement (i.e., "curse of dimensionality") due to dynamic programming. Future work to address this issue includes decomposition of the system [17], using hand-tuned CLF candidates to warm-start the computation [28], adaptive grids [19], and data-driven approaches [18]. Other directions of interest include incorporating disturbances, exploring systems with multiple isolated equilibrium points, finding connections with black-box models, and tuning γ's online to achieve different stabilizability properties.

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