# Emergence of a spontaneous singularity in the optimal guidance of buoyancy-controlled balloons in hurricanes with an absolute value penalty on the control

Paolo Luchini[1], Thomas Bewley[2]
[1]Università di Salerno – DIIN, 84084 Fisciano (SA), Italy
[2]Flow Control Lab, Dept of MAE, UC San Diego, La Jolla CA, USA

January 5, 2025

**Abstract**

The optimal control of a linear system disturbed by white noise with a quadratic objective leads to the classical ("LQG") linear feedback, with gaussian statistical distributions of all quantities involved. Shifting to a non-quadratic objective, as considered in this work, removes many convenient mathematical properties in this formulation; in particular, the optimal controller can no longer be assumed to be linear, even though the system to be controlled is linear, and the associated statistical distributions are non-gaussian. In this more general case, the statistical distribution of the state must be modeled over phase space using a dynamic programming approach leveraging a Fokker-Planck PDE.

This work considers a simplified model of the motion of a balloon in a hurricane with strong vertical stratification of the horizontal wind, modelled as a linear system $dx/dt = v + n_v$, $dv/dt = u + n_a$, with state $\{x, v\}$ representing the relative horizontal {position,velocity} of the balloon disturbed by white noise with spectral densities $\{\kappa, \epsilon\}$ (modeling the diffusivity and dissipation of the turbulence), with control $u$ representing the vertical velocity of the balloon (associated with its inflation/deflation within the vertically-stratified horizontal flow environment), and with cost $J = \langle x^2 \rangle + l^2 \langle |u|^c \rangle$. The classical LQG case, with $c = 2$, is considered first, to verify the correctness of our Fokker-Planck solver; we then gradually reduce $c$ to 1 to focus on the problem of physical interest here, in which the relevant penalty on the control is proportional to the average absolute value of the vertical velocity. Numerical solution of this problem leads to a perhaps unexpected discontinuous feedback control rule, with $u = 0$ over much of phase space, which turns out to be particularly convenient from an implementation perspective.

$$dx/dt = u + n_x, \qquad \langle n_x(t) n_x(t')\rangle = N_x \delta(t-t'), \qquad J = \left\langle x^2 \right\rangle + l^2 \left\langle |u|^c \right\rangle$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \frac{\partial (uf)}{\partial x} = \frac{N_x}{2} \frac{\partial^2 f}{\partial x^2}.$$

$$\begin{aligned} dx/dt &= u + n_x, & \langle n_x(t) n_x(t')\rangle &= N_x \delta(t-t'), \\ du/dt &= y + n_u, & \langle n_u(t) n_u(t')\rangle &= N_u \delta(t-t'), \end{aligned} \qquad J = \left\langle x^2 \right\rangle + l^2 \left\langle |y|^c \right\rangle$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \frac{\partial (uf)}{\partial x} + \frac{\partial (yf)}{\partial u} = \frac{N_x}{2} \frac{\partial^2 f}{\partial x^2} + \frac{N_u}{2} \frac{\partial^2 f}{\partial u^2}.$$

$$\begin{aligned} dx/dt &= u + n_x, & \langle n_x(t) n_x(t')\rangle &= N_x \delta(t-t'), \\ du/dt &= y + n_u, & \langle n_u(t) n_u(t')\rangle &= N_u \delta(t-t'), \\ dy/dt &= v + n_y, & \langle n_y(t) n_y(t')\rangle &= N_y \delta(t-t'), \end{aligned} \qquad J = \left\langle x^2 \right\rangle + l^2 \left\langle |v|^c \right\rangle$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \frac{\partial (uf)}{\partial x} + \frac{\partial (yf)}{\partial u} + \frac{\partial (vf)}{\partial y} = \frac{N_x}{2} \frac{\partial^2 f}{\partial x^2} + \frac{N_u}{2} \frac{\partial^2 f}{\partial u^2} + \frac{N_y}{2} \frac{\partial^2 f}{\partial y^2}.$$

$$\begin{aligned} dx/dt &= u + n_x, & \langle n_x(t) n_x(t')\rangle &= N_x \delta(t-t'), \\ du/dt &= y + n_u, & \langle n_u(t) n_u(t')\rangle &= N_u \delta(t-t'), \\ dy/dt &= v + n_y, & \langle n_y(t) n_y(t')\rangle &= N_y \delta(t-t'), \\ dv/dt &= a + n_v, & \langle n_v(t) n_v(t')\rangle &= N_v \delta(t-t'), \end{aligned} \qquad J = \left\langle x^2 \right\rangle + l^2 \left\langle |a|^c \right\rangle$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \frac{\partial (uf)}{\partial x} + \frac{\partial (yf)}{\partial u} = \frac{N_x}{2} \frac{\partial^2 f}{\partial x^2} + \frac{N_u}{2} \frac{\partial^2 f}{\partial u^2}.$$

# 1 Introduction and previous work

The forecasting of hurricanes can benefit significantly from GPS-equipped sensor-laden balloons that can be guided to distribute over, and persist within, the flowfield of interest (evolving over a large geographic region), for days at a time, rather than a dozen minutes or so as is the case with today's free-flying balloons and dropsondes. A promising energy-sparing technology consists of balloons with mechanically-adjustable volume [1] which can reversibly provide increased or decreased buoyancy and move to an altitude where the prevailing wind blows in the desired direction. In previous work we have shown that a) guidance of balloons in specified orbits, and even in orderly formations, can numerically be achieved in realistic hurricane simulations [2], and b) a control objective proportional to the absolute value of the inflation rate, more representative of required electric power than its square, leads to the choice of a discontinuous control law where the balloon is left most of the time at a constant volume and only inflated or deflated for abrupt short periods (ideally, in a discontinuous way) [3, 4].

In this context, result (b) mentioned above was achieved by simplifying the dynamics, to the point that the entire turbulent flow was replaced by a white noise with its spectral amplitude as the only tunable parameter. A more realistic approximation must at least involve a spectrum that more closely resembles a turbulent flow. It is an almost forgotten result that the lagrangian correspondent of the Kolmogorov $k^{-5/3}$ spatial spectrum of turbulent energy is a temporal spectrum proportional to $\omega^{-2}$ [5]. An $\omega^{-2}$ power spectrum implies an $\omega^{-1}$ amplitude spectrum, and is quite easy to achieve in a lumped numerical simulation by passing white noise through an integrator.

# 2 Spectra of lagrangian tracers in turbulence

Kolmogorov's classical theory of homogeneous isotropic turbulence is based on the assumption that energy "flows" between adjacent scales of length at a constant rate until it reaches scales so small that it is eventually dissipated by viscosity. Its basic assumption is that there exists an energy flux (or cascade) in wavenumber space, *i.e.* that energy flows "locally" from wavenumber $k$ to wavenumber $2k$ and from $2k$ to $4k$, but with no jumping the chain. In a sense this is implicit in the quadratic structure of the Navier-Stokes equations. Energy in this statement must be read as kinetic energy per unit mass of fluid, half the square of fluid velocity with the dimensions of $m^2/s^2$, and we shall assume "per unit mass" as understood wherever we speak of just energy in what follows. Since the average amount of viscous energy dissipation (in a statistically steady state) is a power $\varepsilon$ with the dimensions of energy per unit time, or $m^2/s^3$, it ensues that the energy flux at any scale (larger than the dissipation scale) equals $\varepsilon$. Kolmogorov's spectrum then follows by dimensional analysis: the wavenumber spectrum $S_k(k)$ is a function of the modulus only of the wavenumber vector $\underline{k}$, measured in $m^{-1}$, and has the dimensions of energy per unit wavenumber, or $m^2/s^2/m^{-1} = m^3/s^2$; the only dimensionally consistent spectrum that can be constructed using $\varepsilon$ and $k$ itself is then

$$\underset{[m^3/s^2]}{S_k(k)} = K_k \underset{[m^2/s^3]^{2/3}}{\varepsilon^{2/3}} \underset{[m]^{5/3}}{k^{-5/3}},$$

where the dimensionless number $K_k$ is universally known as Kolmogorov's constant.

It is not an as frequently applied result, despite being mentioned in Tennekes & Lumley [**?**], that a closely similar dimensional argument applies to the frequency spectrum. If we again assume that all the important physical characteristics at inertial scales are resumed in the single parameter of energy dissipation (or, perhaps more appropriately, energy flux although it amounts to the same number) $\varepsilon$, the frequency spectrum $S_\omega(\omega)$ has the dimensions of energy per unit frequency, or $m^2/s$, and can be expressed in a unique way as a function of $\varepsilon$ and frequency itself as

$$\underset{[m^2/s]}{S_\omega(\omega)} = K_\omega \underset{[m^2/s^3]}{\varepsilon} \underset{[s]^2}{\omega^{-2}} \tag{1}$$

where $K_\omega$ is another dimensionless constant.

Whereas in the presence of a flow with nonzero mean velocity $V$ an eulerian observer fixed in the laboratory frame will tend to measure a frequency spectrum related to the wavenumber spectrum as $S_\omega \simeq V^{-1} S_k(\omega V^{-1}) \approx \omega^{-5/3}$ (Taylor's hypothesis, see also [5]), a lagrangian observer transported with the fluid (or an eulerian observer in the absence of mean velocity) will measure $S_\omega = K_\omega \, \varepsilon \, \omega^{-2}$. The transition may not be very easy to detect, because the exponents $-5/3$ and $-2$ are relatively close to each other, but nevertheless $-2$ is the correct exponent for the spectrum seen by a lagrangian observer, and this realization will provide a greatly simplified dynamical model for the purposes of the next section.

# 3 Representative dynamical system of a lagrangian balloon

A meteorological balloon in a hurricane is $10^3$ times smaller than the transversal (vertical) dimension of the hurricane itself, that is in the same size ratio as a $10\mu m$ particle in a glass of water. Both can be mechanically treated as point particles in a locally homogeneous and isotropic turbulent field, with the additional advantage that the balloon sits flatly in the inertial part of the turbulent spectrum. On a sufficiently slow time scale (compared to the time constant given by the ratio of its mass and aerodynamic resistance), the balloon will be simply transported with the local instantaneous velocity of the fluid. Namely, it will obey the equation of motion

$$\frac{\mathrm{d}\underline{\mathrm{x}}}{\mathrm{d}t} = \underline{\mathrm{V}}(\underline{\mathrm{x}}) + \underline{\mathrm{v}}$$

where $\underline{\mathrm{V}}(\underline{\mathrm{x}})$ is the mean and $\underline{\mathrm{v}}$ the fluctuating (stochastic) component of the fluid's velocity. More precisely this equation will be obeyed by the horizontal components of the balloon's position vector, as we assume that the balloon can control its vertical position by changing its buoyancy. In an extreme simplification, but one that captures the relevant energetic balance, we can state that the vertical position $z$ is given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = u$$

where $u$ is proportional to the gas flow rate with which the pump inflates the balloon (or more realistically but equivalently, the rate at which a winch changes its volume).

In order to maintain the balloon in a circular orbit around the eye of the hurricane we want to control a single component of $\underline{\mathrm{x}}$ representing its radial position. Denoting this component as simply $x$, in [4] we assumed that the mean horizontal velocity would be a linear function of vertical position, $V = gz$ for some velocity gradient $g$, and the fluctuating velocity a white noise $v = n_v$ since this is the simplifying assumption that is standard in optimal control theory. This setup led to the dynamical system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = u \tag{2a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = gz + n_v, \tag{2b}$$

with the autocorrelation of $n_v$ given by $\langle n_v(t)n_v(t')\rangle = N\delta(t-t')$ and its spectrum $S_\omega = N$ $(m^2/s)$.

In the light of §2 we can now assume a more realistic turbulent fluctuation, one with frequency spectrum $S_\omega = \varepsilon\,\omega^{-2}$ (We can for the present purposes absorb the dimensionless constant $K_\omega$ of (1) into $\varepsilon$.) To do so is actually quite easy, because by the standard rules of linear-filter theory the spectrum of the output of a filter is the spectrum of its input multiplied by the squared modulus of its response function. Therefore a spectrum proportional to $\omega^{-2}$ is quickly synthetized by passing a white noise (constant spectrum) through a filter with response $(\mathrm{i}\omega)^{-1}$, namely an integrator. In formulas

$$\frac{\mathrm{d}v}{\mathrm{d}t} = n_a, \tag{3}$$

with $\langle n_a(t)n_a(t')\rangle = \varepsilon\delta(t-t')$. What (3) shows is that in (the inertial range of) a homogeneous turbulent flow, where the spectrum of velocity is far from constant, the lagrangian spectrum of *acceleration* is actually the constant one of a white noise.[1]

[1] A spectrum having a low-wavenumber cutoff, and thus a finite integral time scale, is just as easy to synthetize by replacing (3) with the damped integrator

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\sigma v + n_a,$$

which produces the spectrum $S_\omega = \varepsilon\,(\omega^2+\sigma^2)^{-1}$ and the finite variance $\langle v^2\rangle = \int_{-\infty}^{\infty} S_\omega\,\mathrm{d}\omega/(2\pi) = \varepsilon/(2\sigma)$. In practice the actual shape of a turbulent spectrum at low frequency will change from case to case, and most likely homogeneity and isotropy will be lost at that scale as well.

Equations (2,3) can be combined together in a more compact form by redefining $v \leftarrow g\,z + v$ (the local velocity of the balloon and the fluid at height $z$, rather than the velocity at the reference height $z = 0$) and $u \leftarrow g\,u$. We then obtain

$$\frac{\mathrm{d}v}{\mathrm{d}t} = u + n_a \tag{4a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v. \tag{4b}$$

Before delving into the control aspects of (4) it may be interesting to highlight some qualitative properties of the uncontrolled system, that is a lagrangian tracer (balloon) abandoned to a wiggling turbulent flow. Our previous model (2), for zero control $u$, can be recognized as the Langevin equation of brownian motion [**?**]. When $v$ is a white noise, $x$ is a brownian motion, just as is the case in the canonical experiment with microscopic particles in thermal agitation. In brownian motion, position $x$ never attains a statistically steady state (or it could be said that its statistical mean is infinite, which corresponds to the observation that a brownian particle wanders away indefinitely if given enough time), and if localized at a single point $x = 0$ at time $t = 0$, its variance subsequently grows linearly in time as $\langle x^2 \rangle = 0.5Nt$. The more realistic equation (3) shows that in a homogeneous turbulent flow *velocity* itself is a brownian motion, and its statistical mean is infinite whereas it variance grows indefinitely in time (in a practical turbulent flow only until the integral time scale is attained, but still this is a consideration with possibly nontrivial implications, showing that the existence of a mean velocity is not always to be taken for granted). The variance of position of a balloon localized at a single point $x = 0$ at time $t = 0$, and released with zero initial velocity, grows cubically in time according as $\langle x^2 \rangle \propto \varepsilon\, t^3$, $\varepsilon$ being the energy dissipation (per unit mass of fluid) characteristic of the turbulent flow involved. Notice that this different behaviour is consistent with dimensional analysis, since the dimensions of the dissipation $\varepsilon$, $m^2/s^3$, are different from the dimensions of the velocity white noise $N$ (proportional to temperature) intervening in classical brownian motion, which are those of a diffusion coefficient $m^2/s$.

# 4 LQR optimal control

In fact the lack of an upper bound on the growth of $x$ with time becomes irrelevant once a feedback controller is introduced, which is designed to keep the ballon in proximity of a reference radial position (which for us will be $x = 0$). If the controller is any effective, the statistical average of $x$ will necessarily become finite (as will always be the case in the following examples), and whether it would be infinite or just much larger had the controller been absent makes no difference.

Nevertheless a word of caution is needed concerning the different ways of writing velocity $v$ in (4) and (3). In (4) $v$ is the actual balloon velocity $v(z)$

as measured at its current height $z$, for instance from GPS, and in this sense certainly more tangible as an input to the controller than the unmeasurable fluid velocity $v(0)$ at the reference height $z = 0$. On the other, the fluid velocity $v(0)$ at the reference height $z = 0$ is free to wander to infinity even when $v(z)$ is being kept finite by the controller. What this in fact means is that the wind can in principle attain an unbounded velocity (because we are not imposing any low-frequency cut-off or equivalently a finite integral time scale), but it does so slowly enough that the controller can always compensate it by an equally unbounded (again, in principle) change of altitude $z$. In practice none of these quantities will be infinite, but we can expect balloon altitude $z$ to fluctuate in a wide range. By allowing this range to be infinite we gain the possibility to set up the control model as a statistically stationary problem without having to take an additional time scale into account.

Optimal control is particularly relevant to the design of such a controller because energy onboard the balloon (electrical energy, not to be confused with the turbulent fluctuation energy that has been considered so far) will be strongly limited, and therefore more than in other cases we want to achieve control of the trajectory with the least possible power expenditure.

In order to compare the present results to those of [4], we can formulate a hybrid problem which encompasses both. By writing the dynamical system with both an acceleration noise and a velocity noise, as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = u + n_a \tag{5a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v + n_v. \tag{5b}$$

it should be evident enough that for $n_v = 0$ we reobtain (4) while for $n_a = 0$ (and $u \leftarrow gu$), we reobtain (2). More generally, $n_v$ will be a white noise with correlation function $\langle n_v(t)n_v(t')\rangle = N\delta(t-t')$, and $n_a$ a white noise with correlation function $\langle n_a(t)n_a(t')\rangle = \varepsilon\delta(t-t')$. Equation (5) is in fact the representation of a fairly general second-order mechanical system (a double integrator) subjected to white-noise disturbances in all its state components, a hallmark textbook example of control theory.

In order to define (5) as a control problem, we have to specify an objective function to be minimized. Typically we want to maintain the balloon as close as possible to the origin, say we want to minimize the variance of its position, with a constraint on spent power. In the simplest assumption the latter is proportional to the square of the inflating pump's flow rate $u$, and by imposing the constraint through a Lagrange multiplier $l^2$ we get the objective function

$$J = \left\langle x^2 \right\rangle + l^2 \left\langle u^2 \right\rangle. \tag{6}$$

Equivalently we may choose to minimize spent power for a given target variance of the position $x$, which leads to the exact same objective function.

The problem of minimizing a quadratic objective function for a linear dynamical system, the Linear Quadratic Regulator problem as it is commonly denoted, has a number of useful mathematical properties that allow its solution to prescind almost completely from its stochastic character. Minimizing (6) for the system (5) is nowadays a classroom control problem, but unfortunately its favourable properties are lost as soon as one of the assumptions is removed, as will be the case in the next section. Therefore it may be useful to write down the procedure that leads to the LQR solution as a comparison.

In particular, for an LQR problem we know that the optimal controller is a linear controller, where $u$ is a linear function of the state variables [**?**]. Linearity of both the original and the closed-loop problem implies that if the excitation is a white (or just gaussian) stochastic process every other statistics is gaussian as well, and that the forced problem with a stochastic excitation is equivalent to (can be written as a linear combination of) a set of deterministic problems with independent forcing vectors. The latter in turn are equivalent to a set of initial-value problems with independent initial-condition vectors; as a consequence the usual textbook formulation of LQR only involves optimization of an initial-value problem.

# 5 Optimal control with a non-quadratic cost

Remembering our aim to keep the balloon on its trajectory with the least amount of electric power, we may want to more closely re-examine how this power is estimated. When a pump inflates a balloon it has to do work against the pressure difference that exists between the inside and the outside. Spent power would be proportional to the square of gas flow rate $u$ if this pressure difference were proportional to $u$ itself, as would be the case, for instance, in low-Reynolds number flow through a pipe, but this is hardly realistic for a balloon. More likely the pressure difference depends on the total volume of contained gas, and on the elastic properties of the envelope, but very little (within reasonable margins) on flow rate. In a schematization opposite to the one assumed in the previous section, we may roughly assume the pressure difference as being a constant; work done by the pump is then *linearly* proportional to $u$. More precisely, in a cycle of inflating and deflating the pump only does work when inflating, whereas when the ballon is deflated energy is lost (unless some energy-recovery device is adopted that for now we shall consider absent); therefore we must sum the increases of $u$ when $u$ is positive only. However, in a statistically stationary process the sum of the positive increases of $u$ exactly balances the sum of its negative decreases; therefore the average of positive increases is half the average of absolute values. Eventually the objective to be minimized becomes the statistical expectation of the absolute value of gas flow rate, $\langle |u| \rangle$, or once this is combined with the variance of $x$ through a Lagrange multiplier,

$$J = \left\langle x^2 \right\rangle + l^2 \left\langle |u| \right\rangle . \tag{7}$$

The seemingly marginal difference between (6) and (7) all of a sudden makes LQR theory inapplicable. Even when the open-loop system is linear, it can no longer be proved that its optimal controller is linear as well, and generally with a non-quadratic objective it cannot be expected to be so. Once the closed-loop system becomes nonlinear the stochastic process becomes nongaussian, and the stochastic control problem can no longer be reduced to a deterministic initial value problem and must be studied in its entirety.

### 5.1 Fokker-Planck equation

Upon introducing the probability density $f(x, v, t)$, such that $f\mathrm{d}x\mathrm{d}v$ represents the probability that the position and velocity of the balloon are in the interval $[x, x + \mathrm{d}x; v, v + \mathrm{d}v]$ at time $t$, the statistics of (5) are governed by the two-dimensional Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \frac{\partial(vf)}{\partial x} + \frac{\partial(uf)}{\partial v} = \frac{N}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\varepsilon}{2}\frac{\partial^2 f}{\partial v^2}. \tag{8}$$

Now the control $u = u(x, v)$ can be a general nonlinear function, which we want to determine in such a manner that the objective (7) is minimized. The steady solution, which we shall mostly be interested in, is simply obtained by setting the derivative $\partial f/\partial t = 0$. Notice that (8) is a homogeneous problem that has a nontrivial solution, and must generally be accompanied by the normalization condition

$$\int f\,\mathrm{d}x\,\mathrm{d}v = 1 \tag{9}$$

In order to encompass both cases of (6) and (7) (and intermediates between them), we can more generally write the objective function as

$$J = \int \left[x^2 + l^2 w(u)\right] f(x, v, t)\,\mathrm{d}x\,\mathrm{d}v \tag{10}$$

where $w(u)$ is a given function of one variable that we may specify later, and the averaging has been made explicit as an integral over the probability distribution. Adding the differential equation (8), which acts as a constraint, through a Lagrange multiplier $p(x, v)$, and the normalization condition (9) through another Lagrange multiplier $\lambda$, gives the variational problem

$$\delta J = \delta \int \Big\{ \left[x^2 + l^2 w(u) - \lambda\right] f + \\ + p\left[v\frac{\partial f}{\partial x} + \frac{\partial(uf)}{\partial v} - \frac{N}{2}\frac{\partial^2 f}{\partial x^2} - \frac{\varepsilon}{2}\frac{\partial^2 f}{\partial v^2}\right]\Big\}\,\mathrm{d}x\,\mathrm{d}v = 0.$$

By the usual procedure of integration by parts the optimality conditions, or adjoint equations, can now be worked out as

$$v\frac{\partial p}{\partial x} + u\frac{\partial p}{\partial v} + \frac{N}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\varepsilon}{2}\frac{\partial^2 p}{\partial v^2} = x^2 + l^2 w(u) - \lambda \tag{11a}$$

$$l^2\frac{\mathrm{d}w}{\mathrm{d}u} = \frac{\partial p}{\partial v}. \tag{11b}$$

Since not only the system equation (8) like all Fokker-Planck equations is linear in $f$, but contrary to what happens in LQR control also the objective (10) is, the direct and adjoint problems are decoupled. Curiously the value of the Lagrange multiplier $\lambda$ coincides with the value of the objective function $J$ itself. In fact, just as the direct equation is a homogeneous problem which has a nontrivial solution, its adjoint (11a) is a nonhomogeneous problem which only has a solution at all under a compatibility condition. This condition is that its *r.h.s.* be orthogonal to $f$, *i.e.*

$$\int \left[x^2 + l^2 w(u) - \lambda\right] f \, \mathrm{d}x \, \mathrm{d}v = 0.$$

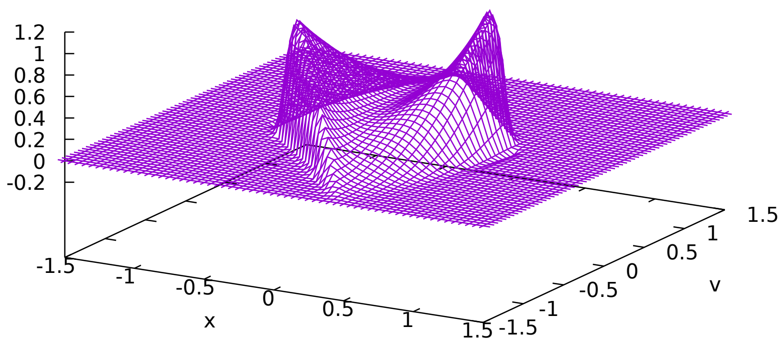
Combining the last result with (10) and (9) now gives $J = \lambda$. This relationship provides a means to obtain the value of $J$ from the solution of (11) without ever computing $f$. (Although computing $f$ can be interesting in its own right.)

The solution to equations (8) and (11) can be sought for numerically. It helps in this respect to consider that the form of these equations is mathematically identical to those of scalar transport in a recirculating fluid flow, and the relevant numerical techniques can be directly borrowed from fluid dynamics. This analogy also helps in recognizing the appropriate boundary conditions at infinity: (8) represents an incoming flow and needs a condition of zero incoming particle flux in order to make the total probability constantly equal to 1 even in a discrete setting; (11a) represents an outgoing flow and will give a result almost independent of which boundary condition is imposed except in the near proximity of the artificial boundary itself. It must also be remarked that, in the case where the function $w$ is absolute value, its derivative which appears in (11b) is the sign function and is discontinuous (piecewise constant). We must expect therefore that this case may develop peculiar behaviour; it helps in this respect to consider a family of problems defined by

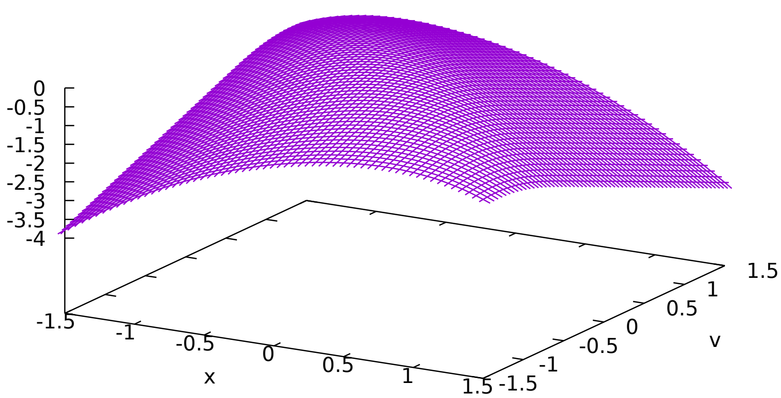
$$w(u) = |u|^c,$$

which reduces with continuity to (6) for $c = 2$ and to (7) for $c \to 1$.

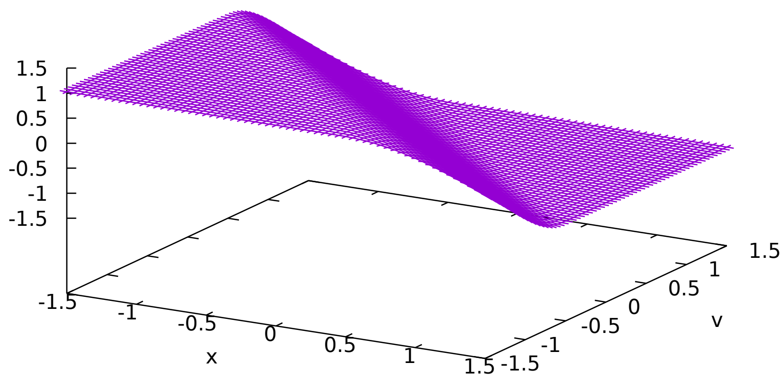
## 6 Numerical results



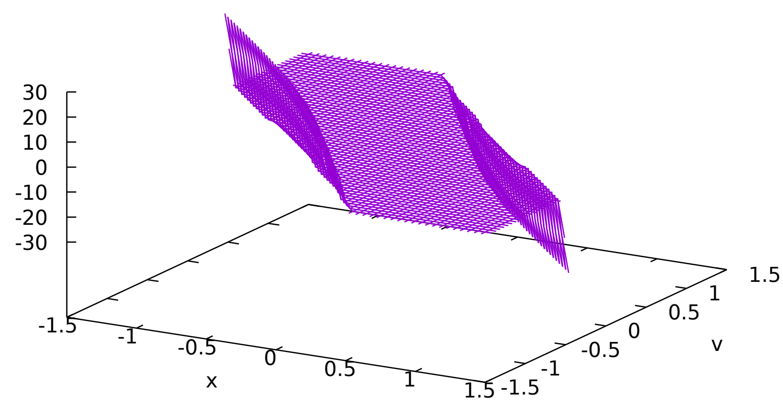

$c = 1$, probability distribution.



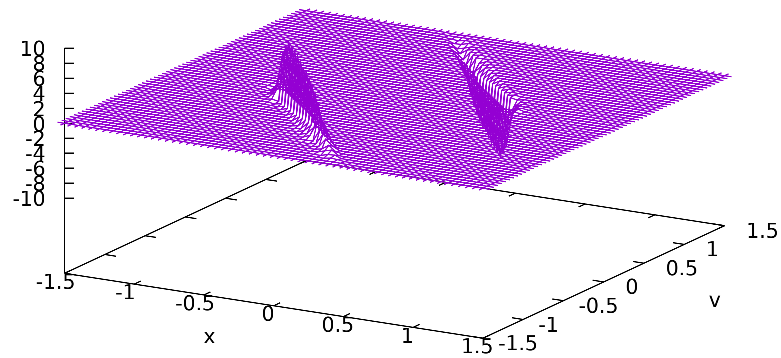

$c = 1$, adjoint function.



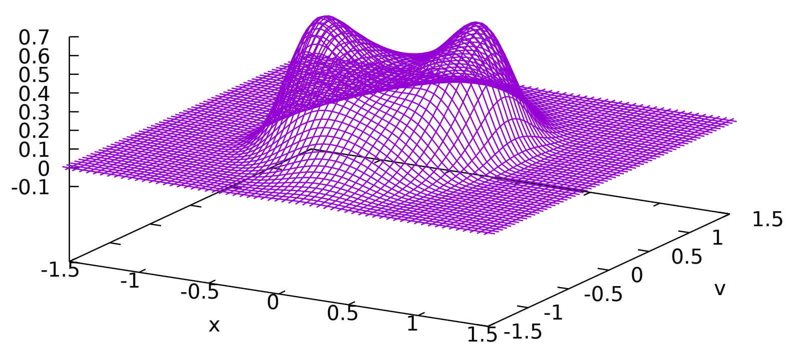

$c = 1$, adjoint function derivative.



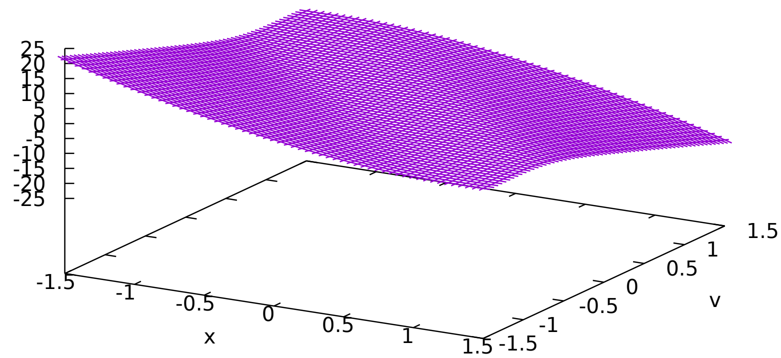

$c = 1$, control function.



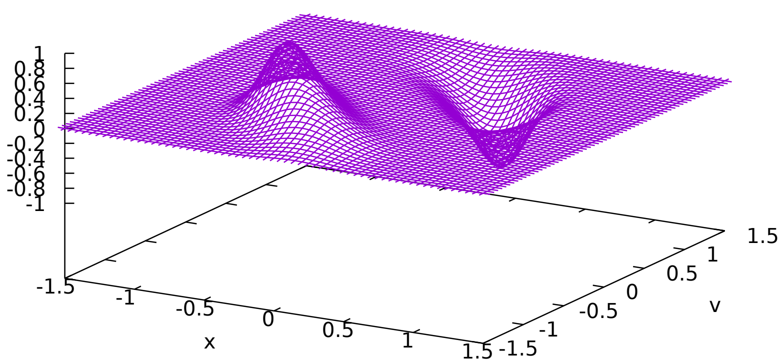

$c = 1$, effective control.



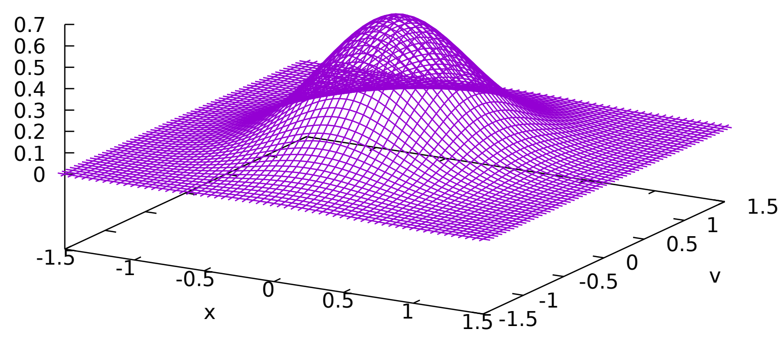

$c = 1.2$, probability distribution.



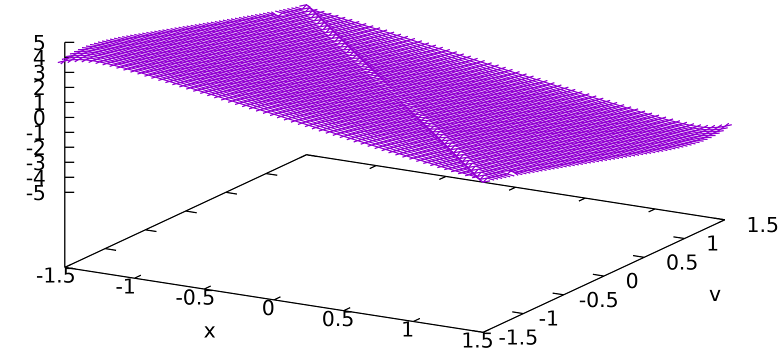

$c = 1.2$, control function.



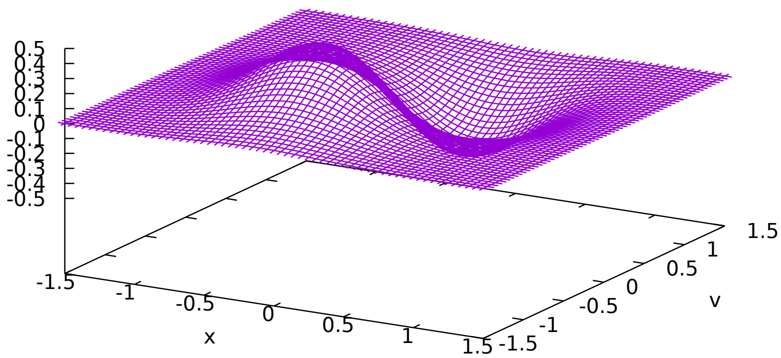

$c = 1.2$, effective control.




$c = 2$, probability distribution.




$c = 2$, control function.




$c = 2$, effective control.

### 6.1 Interpretation of the discontinuous solution

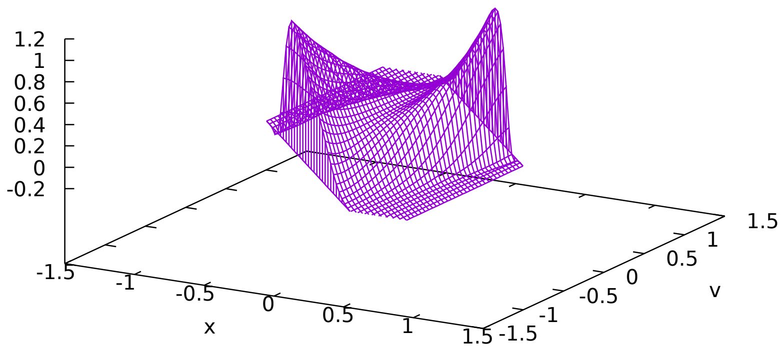
The control function obtained when spent power is proportional to the absolute value of $u$ has a clearly discontinuous appearance, with a central plateau where $u = 0$ bounded by two nearly straight oblique lines where it suddenly jumps to very high positive or negative values. Such discontinuous jumps act as impenetrable barriers, which reject particles (balloons) attempting to pass them. This is a very interesting form of on-off control, interesting because it is very often adopted even when it is not optimal for the practical reason that an on-off controller is easier to build than a continuous controller. In the present application where power spent is at a premium, knowing that an on-off controller is actually the optimal one gives a double advantage. In addition, an on-off controller really consumes no power during the periods it is off.

To verify that a discontinous on-off controller is really what the numerical simulations of the previous section ar aiming towards, we set up a separate simulation where a barrier is imposed, in the form of a zero-flux boundary condition, at the position of a priori unspecified straight line, and the location of this straight line is optimized for. Since in the interior we assume $u = 0$, (8) reduces to

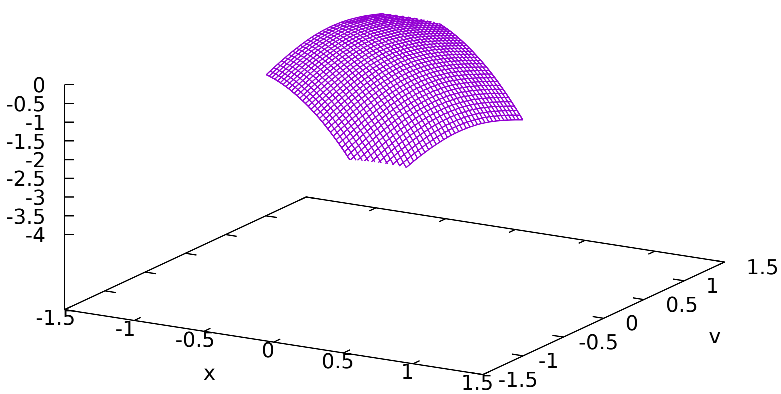
$$\frac{\partial(vf)}{\partial x} = \frac{\varepsilon}{2}\frac{\partial^2 f}{\partial v^2},$$

and (11a) to

$$v\frac{\partial p}{\partial x} + \frac{\varepsilon}{2}\frac{\partial^2 p}{\partial v^2} = x^2 - \lambda.$$




discontinuous probability distribution.




discontinuous adjoint function.

# 7 Practical application

Once the problem is brought to the form of (5), the answer to the linear-quadratic optimal control problem appears in its surprising simplicity: LQ control does not care whether noise is in one or the other state equation or both, as long as it is white. Therefore

> The optimal linear controller for the Kolmogorov-noise ballon problem is identical to the optimal controller for the white-noise ballon problem.

This is astounding, given the large difference in turbulence statistics. Just as in last year's notes, in either case the controller is

$$gu = -2\sigma v - 2\sigma^2 x$$

$\sigma$ being a "strength" (or gain) parameter indirectly determined by the variance of $x$ one is aiming for and the related cost.

Whereas the optimal controller is the same, the obtained optimum is different. The impulse response of (5) corresponding to $w_\varepsilon$ excitation is obtained from the initial-value problem $x(0) = 0$; $v(0) = 1$ (as opposed to $x(0) = 1$; $v(0) = 0$ for $w_\nu$ excitation) and is

$$x = \frac{1}{2\mathrm{i}\sigma}\left[\mathrm{e}^{-\sigma(1-\mathrm{i})t} - \mathrm{e}^{-\sigma(1+\mathrm{i})t}\right]$$

whence

$$\left\langle x^2 \right\rangle = \varepsilon \int_0^\infty x^2\, dt = \frac{\varepsilon}{8\sigma^3}, \quad g^2 \left\langle u^2 \right\rangle = \varepsilon \int_0^\infty u^2\, dt = 2\varepsilon\sigma$$

and, upon eliminating $\sigma$,

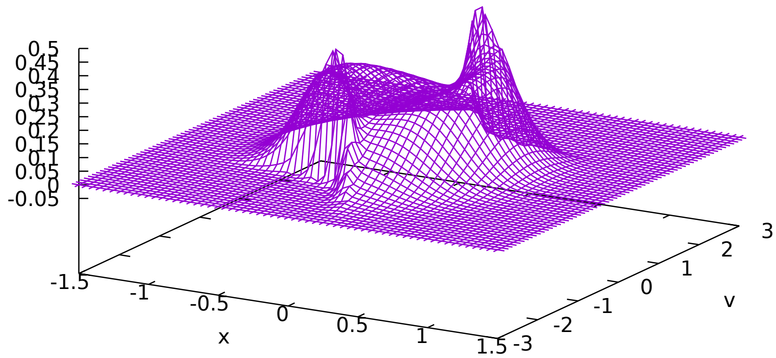
$$g^2 \left\langle u^2 \right\rangle = \varepsilon^{4/3} \left\langle x^2 \right\rangle^{-1/3}$$

(to be compared to $g^2 \left\langle u^2 \right\rangle = (27/64)\nu^4 \left\langle x^2 \right\rangle^{-3}$ for the case of diffusion noise).

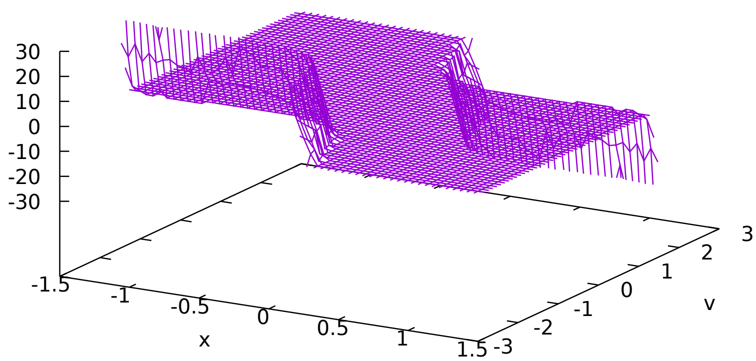
# 8 More general mechanical system

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -a\,x - b\,v + u + n_a \tag{12a}$$

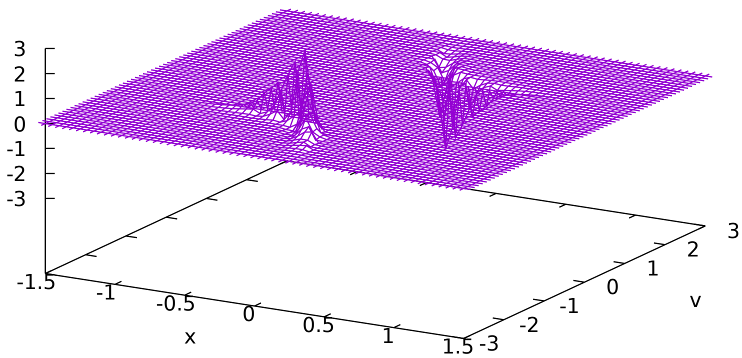
$$\frac{\mathrm{d}x}{\mathrm{d}t} = v. \tag{12b}$$



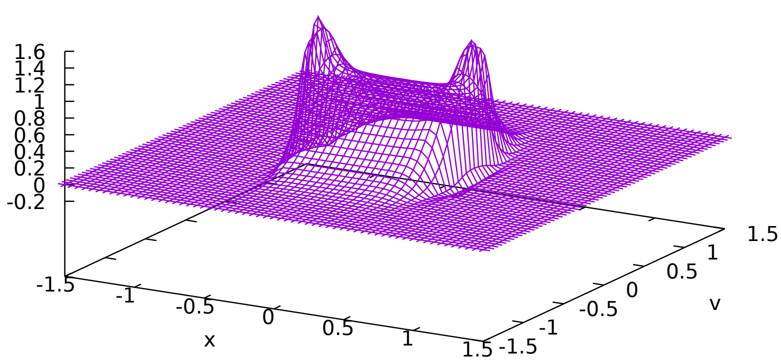

$c = 1$, $a = 4$, probability distribution.



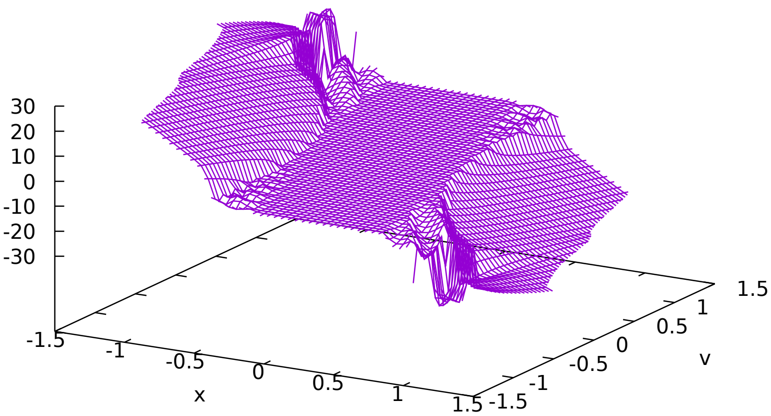

$c = 1$, $a = 4$, control function.



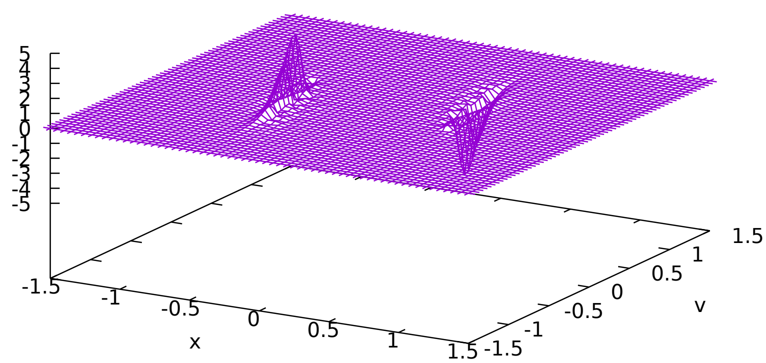

$c = 1$, $a = 4$, effective control.




$c = 1$, $b = 4$, probability distribution.




$c = 1$, $b = 4$, control function.




$c = 1$, $b = 4$, effective control.

# References


[1] G. Meneghello, T.R. Bewley, M. de Jong, and C. Briggs. A coordinated balloon observation system for sustained in-situ measurements of hurricanes. *IEEE Aerospace Conference*, Big Sky. 2017

[2] T.R. Bewley and G. Meneghello. Efficient coordination of swarms of sensor-laden balloons for persistent, in situ, real-time measurement of hurricane development, *Phys. Rev. Fluids* 1, 060507, 2016.

[3] G. Meneghello, P. Luchini, and T.R. Bewley On the control of buoyancy-driven devices in stratified, uncertain flowfields, *International Symposium on Stratified Flows (ISSF)*, San Diego. 2016.

[4] G. Meneghello, P. Luchini, and T.R. Bewley. A Probabilistic Framework for the Control of Systems with Discrete States and Stochastic Excitation, *Automatica*, to appear. 2017.

[5] H. Tennekes, J.L. Lumley. *A First Course in Turbulence* §8.5, MIT Press, 1972.