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Thomas R. Bewley & Parviz Moin
Center for Turbulence Research
Stanford, California 94305-3030

and

Roger Temam
The Institute for Scientific Computing and Applied Mathematics, Indiana University
Bloomington, Indiana 47405-5701

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Thomas R. Bewley and Parviz Moin
Center for Turbulence Research, Stanford University, Stanford, California 94305-3030
and
Roger Temam
The Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405-5701, and
Laboratoire d'Analyse Numérique, Université de Paris-Sud, 91405 Orsay, FRANCE

Abstract

Recent work by the authors in the application of *optimal* control theory to turbulence have been quite successful when full state information is provided to the control algorithm. However, this approach has not yet been successful for the development of algorithms which depend on wall information only. For this reason, *robust* control theory, which is currently well developed only for linear problems, is now examined as a technique by which effective control algorithms based on limited noisy observations might be developed for turbulent flows and other nonlinear phenomena subjected to external disturbances.

1 Introduction

In its essence, robust control theory boils down simply to Murphy's Law (Bloch 1982) taken seriously:

> If a worst-case system disturbance *can* disrupt a controlled closed-loop system, it *will*!

When designing a robust controller, therefore, one must *plan* on a finite component of the worst-case disturbance aggravating the system, and design a controller which is suited to handle even this extreme situation. A controller which is designed to work even in the presence of the Murphy's Law Worst Case Disturbance (MLWCD) will also be robust to a wide class of other possible disturbances which, by definition, are not as bad as the MLWCD. Thus, the serious issue of finding a robust stabilizing controller is intimately coupled with the equally serious issue of finding the MLWCD.

A framework for applying optimal (§2) and robust (§3) control theories to linear problems (such as early stages of transition, see Bewley, Liu, & Agarwal 1997) is first reviewed in a style fairly consistent with standard control theory. Several good texts are available covering linear control theory, including Doyle *et al.* (1989), Green & Limebeer (1995), Lewis (1995), Zhou, Doyle, & Glover (1996), and Skogestad & Postlethwaite (1996). The presentation here is made strictly in the time domain to facilitate extension of these "standard" linear approaches to nonlinear problems, in which frequency domain techniques are of limited usefulness. A framework for applying optimal control theory to nonlinear problems (such as turbulence, see Abergel & Temam 1990 and Bewley, Moin, & Temam 1997) is then reviewed (§4) in a similar notation. Finally, a straightforward connection is made such that the concepts of robust control theory may be extended to nonlinear problems in a consistent manner (§5). The resulting development is straightforward and does not assume the reader is accustomed to the language of control theory, rather, only to the precepts of Murphy's Law.

2 Optimal linear regulation

2.1 State equation

Consider a state vector $\mathbf{u}$ which is a function of some feedback control vector Φ such that it obeys the linear evolution equation

$$\boxed{\dot{\mathbf{u}} = A\,\mathbf{u} + B_2\,\Phi} \tag{1a}$$

with given initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \text{at } t = 0. \tag{1b}$$

The matrices A and B_2 may be functions of time but do not themselves depend on the state $\mathbf{u}$ or the control Φ. In fluid mechanics, such a system may be determined by discretization of a set of PDE's representing a linearized equation of motion for the flow, and thus the system may be expected to be of very high dimension.

2.2 Cost function

The object of applying control in the present problem is to "regulate", or drive to zero, some measure of the state without applying excessive amounts of control.

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Mathematically, this objective is expressed as the minimization of a cost function which balances a measure of the state $\mathbf{u}$ with a measure of the control Φ applied. We will use the norm symbol to denote these measures, to be defined appropriately for specific problems:

$$\mathcal{J}_2 \equiv \frac{1}{2T} \int_0^T \left(||\mathbf{u}||^2 + \ell^2 \, ||\Phi||^2 \right) \, dt.$$

Note that the two terms are weighted together with a positive factor ℓ^2 which accounts for the "price" of the control. This factor is large if applying the control is "expensive", which emphasizes the importance of the latter term in this equation and results in the minimum control effort necessary to stabilize the system, and small if applying the control is "cheap", which results in a larger control effort and thus faster regulation of the nominal, undisturbed plant. The expression is averaged over some optimization interval under consideration $[0, T]$. In matrix form, $\mathcal{J}_2$ is expressed as

$$\boxed{\mathcal{J}_2 = \frac{1}{2T} \int_0^T \left(\mathbf{u}^* \, C_1^* \, C_1 \, \mathbf{u} + \ell^2 \, \Phi^* \, \Phi \right) \, dt}$$

with C_1 defined appropriately based on the definition of $||\mathbf{u}||$ and the star $(^*)$ denoting the conjugate transpose. By appropriate scaling of the vector Φ and the matrix B_2, the norm of the control $||\Phi||$ is taken simply as the Euclidian norm without loss of generality.

A technique to design a feedback control relationship of the form $\Phi = K_2 \, \mathbf{u}$ which minimizes the cost function $\mathcal{J}_2$ is now briefly outlined. The problem of designing controllers based on state estimates when the full state vector $\mathbf{u}$ is not available for feedback is closely related to the problem of designing the full-information controller itself. The discussion presented here will thus focus just on the problem of control and not on the dual problem of state estimation.

2.3 Adjoint equation

Define an adjoint state (as yet, arbitrarily) by the relation

$$-\dot{\lambda} = A^* \, \lambda + C_1^* \, C_1 \, \mathbf{u} \tag{2a}$$

with initial conditions

$$\lambda = 0 \qquad\qquad \text{at } t = T. \tag{2b}$$

Note that the "initial" conditions (2b) are defined at $t = T$, so to determine the adjoint on the interval $[0, T)$, the evolution equation (2a) must be marched *backwards* from $T \to 0$. Note also that the term $C_1^* \, C_1 \, \mathbf{u}$ chosen to drive this equation is closely related to a quantity of interest in the cost function.

2.4 Gradient of cost with respect to control

It is easy to show that the gradient of the cost function $\mathcal{J}_2$ with respect to the control Φ is a simple function of the appropriately-defined adjoint state given in (2):

$$\frac{\mathscr{D}\mathcal{J}_2}{\mathscr{D}\Phi} = B_2^* \, \lambda + \ell^2 \, \Phi. \tag{3}$$

2.5 Solution of control problem

By (3), the most suitable control which results in

$$\frac{\mathscr{D}\mathcal{J}_2}{\mathscr{D}\Phi} = 0 \qquad \text{(minimum)}$$

as a function of the adjoint state is given simply by

$$\Phi = -\frac{1}{\ell^2} B_2^* \, \lambda \tag{4}$$

Combining the state equation (1a), the adjoint equation (2a), and the control given by (4) into a combined matrix form gives

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -\frac{1}{\ell^2} \, B_2 \, B_2^* \\ -C_1^* \, C_1 & -A^* \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \lambda \end{bmatrix} \tag{5}$$

The large matrix on the RHS is referred to as the Hamiltonian matrix. Now prescribe a general relationship between any state vector $\mathbf{u} = \mathbf{u}(t)$ and the corresponding adjoint $\lambda = \lambda(t)$ such that

$$\lambda = X_2 \mathbf{u}, \tag{6}$$

where $X_2 = X_2(t)$. Inserting this expression into (5) to eliminate λ and combining the top and bottom rows to eliminate $\dot{\mathbf{u}}$ leads to the expression

$$\Big(-\dot{X}_2 = A^* \, X_2 + X_2 \, A - X_2 \, \frac{1}{\ell^2} \, B_2 \, B_2^* \, X_2 + C_1^* \, C_1 \Big) \, \mathbf{u}$$

As this expression is valid for any state vector $\mathbf{u}$, we arrive at a time dependent Riccati equation for $X_2(t)$:

$$-\dot{X}_2 = A^* \, X_2 + X_2 \, A - X_2 \, \frac{1}{\ell^2} \, B_2 \, B_2^* \, X_2 + C_1^* \, C_1 \tag{7a}$$

with initial conditions, due to (2b) and (6), given by

$$X_2 = 0 \qquad\qquad \text{at } t = T. \tag{7b}$$

Combining (4) and (6), the optimal control Φ as a function of the state $\mathbf{u}$ is given by the state feedback relationship

$$\boxed{\Phi = K_2 \, \mathbf{u} \qquad \text{where} \qquad K_2 = -\frac{1}{\ell^2} \, B_2^* \, X_2} \tag{8}$$

where $X_2(t)$ is the solution of the time dependent Riccati equation (7) and thus $K_2 = K_2(t)$.

2.6 Infinite time horizon for time invariant problems

If the matrices A, B_2, and C_1 are time invariant, then in the limit of large optimization intervals $T \to \infty$ the matrix $X_2(t)$ defined by (7) approaches a steady state value in the march from the initial conditions defined at $t = T$ back towards $t = 0$. This steady state value may be found by setting $\dot{X}_2 = 0$ in (7a), which leads to

$$\boxed{0 = A^* X_2 + X_2 A - X_2 \frac{1}{\ell^2} B_2 B_2^* X_2 + C_1^* C_1} \qquad (9)$$

The optimum feedback relationship given by (8) in this limit is thus time invariant and a function of the solution to (9), referred to as an algebraic Riccati equation. Solution methods for equations of this type are well developed (Laub 1991).

2.7 Simple interpretation of the adjoint field

In the preceding discussion, the determination of optimal feedback control relationship $\Phi = K_2 \mathbf{u}$ in (8) was closely linked to the definition of an adjoint λ in (2). However, the definition of λ was made arbitrarily in (2), and subsequently justified only mathematically in (3) as being that field which is required to express the gradient of the cost function with respect to the control $\mathscr{D}\mathcal{J}_2/\mathscr{D}\Phi$ in a simple manner.

In the case that the control Φ enters the state equation (1) through the identity matrix, $B_2 = I$, a simple interpretation of the adjoint is now clear. In this case, the expression for the gradient (3) reduces to

$$\frac{\mathscr{D}\mathcal{J}_2}{\mathscr{D}\Phi} = \lambda + \ell^2 \Phi. \qquad (10)$$

Thus, the gradient consists of two terms. The second term on the RHS of (10) simply accounts for the term in the cost function $\mathcal{J}_2$ which measures the magnitude of the control; in the absence of the other term in the cost function, this term would drive the control to zero when $\mathcal{J}_2$ is minimized.

The first term on the RHS of (10) accounts for the term in the cost function $\mathcal{J}_2$ which measures the state $\mathbf{u}$ itself. Thus, one interpretation of the adjoint λ is simply that:

> The adjoint, when properly defined, is a measure of the sensitivity of the term of the cost function which measures the state to additional RHS forcing of the state equation.

Note that there are exactly as many components of the adjoint λ as there are components of the state equation (1a)—this is not by accident.

3 Robust linear regulation

3.1 State equation

Consider the linear state equation (1) with additional forcing due to an external disturbance χ

$$\boxed{\dot{\mathbf{u}} = A\,\mathbf{u} + B_1\,\chi + B_2\,\Phi} \qquad (11a)$$

with given initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \text{at } t = 0. \qquad (11b)$$

The matrix B_1 may be a function of time but does not itself depend on the state $\mathbf{u}$ or the control Φ.

3.2 Cost function

The object of applying control in the robust problem is identical to the optimal problem, except we now play the "devil's advocate" and seek to find the best control in the presence of a "small" component of exactly that disturbance χ which is maximally aggravating to the control objective, the aforementioned MLWCD. To represent this concept mathematically, we append to the cost function $\mathcal{J}_2$ discussed in §2.2 a term which accounts for the magnitude of the disturbance used to aggravate the system

$$\mathcal{J}_\infty \equiv \frac{1}{2T}\int_0^T \left(||\mathbf{u}||^2 + \ell^2\,||\Phi||^2 - \gamma^2\,||\chi||^2\right)\,dt.$$

Note that the sign of the term which is used to account for the disturbance is opposite to the sign used to account for the control; this is because we *minimize* with respect the control Φ while simultaneously we *maximize* with respect to the disturbance χ. The term involving $-\gamma^2\,||\chi||^2$ limits the magnitude of the disturbance in the maximization with respect to χ as the term involving $\ell^2\,||\Phi||^2$ limits the magnitude of the control in the minimization with respect to Φ. In matrix form, $\mathcal{J}_\infty$ is expressed as

$$\boxed{\mathcal{J}_\infty = \frac{1}{2T}\int_0^T \left(\mathbf{u}^*\,C_1^*\,C_1\,\mathbf{u} + \ell^2\,\Phi^*\,\Phi - \gamma^2\chi^*\,\chi\right)\,dt}$$

By appropriate scaling of the vector χ and the matrix B_1, the norm of the disturbance $||\chi||$ is taken simply as the Euclidian norm without loss of generality.

A technique to design a feedback control relationship of the form $\Phi = K_\infty\, \mathbf{u}$ which minimizes the cost function $\mathcal{J}_\infty$ in the presence of a small component of the worst external disturbance χ forcing the state equation (11) is now briefly outlined. By designing a feedback control rule effective for a state disturbed in this manner, the control rule which is found is effective in the presence of small disturbances of any type, and can be designed with nearly the same nominal performance (*i.e.* performance on the undisturbed system) as the optimal controller discussed in §2.

3.3 Adjoint equation

Define an adjoint state as for the optimal control case by the relation

$$-\dot{\lambda} = A^*\,\lambda + C_1^*\,C_1\,\mathbf{u} \tag{12a}$$

with initial conditions

$$\lambda = 0 \qquad\qquad \text{at } t = T. \tag{12b}$$

3.4 Gradients of cost w.r.t. control and disturbance

In a manner identical to the derivation leading to (3), the gradient of the cost function $\mathcal{J}_\infty$ with respect to the control Φ and the disturbance χ in this problem are simple functions of the adjoint state defined by (12):

$$\frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\Phi} = B_2^*\,\lambda + \ell^2\,\Phi \quad \text{and} \quad \frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\chi} = B_1^*\,\lambda - \gamma^2\,\chi. \tag{13}$$

3.5 Solution of control problem

By (13), the most suitable control and the MLWCD which result in

$$\frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\Phi} = 0 \;\;\text{(minimum)} \;\;\text{and}\;\; \frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\chi} = 0 \;\;\text{(maximum)}$$

are given simply by

$$\Phi = -\frac{1}{\ell^2}\,B_2^*\,\lambda \quad\text{ and }\quad \chi = \frac{1}{\gamma^2}\,B_1^*\,\lambda. \tag{14}$$

Combining the state equation (11a) and the adjoint equation (12a) with the control and disturbance given by (14) into a combined matrix form gives

$$\dot{\begin{bmatrix}\mathbf{u}\\ \lambda\end{bmatrix}} = \begin{bmatrix} A & \dfrac{1}{\gamma^2}\,B_1\,B_1^* - \dfrac{1}{\ell^2}\,B_2\,B_2^* \\ -C_1\,C_1^* & -A^* \end{bmatrix} \begin{bmatrix}\mathbf{u}\\ \lambda\end{bmatrix} \tag{15}$$

Now prescribe a general relationship between any state vector $\mathbf{u}$ and the corresponding adjoint λ such that

$$\lambda = X_\infty \mathbf{u} \tag{16}$$

Inserting this expression into (15) to eliminate λ and combining the top and bottom rows to eliminate $\dot{\mathbf{u}}$ leads to the expression

$$\Big[-\dot{X}_\infty = A^*\,X_\infty + X_\infty\,A + X_\infty\left(\frac{1}{\gamma^2}\,B_1\,B_1^* - \frac{1}{\ell^2}\,B_2\,B_2^*\right)X_\infty + C_1^*\,C_1\Big]\,\mathbf{u}$$

As this expression is valid for any state vector $\mathbf{u}$, we arrive at a time dependent Riccati equation for $X_\infty(t)$:

$$-\dot{X}_\infty = A^*\,X_\infty + X_\infty\,A + X_\infty\left(\frac{1}{\gamma^2}\,B_1\,B_1^* - \frac{1}{\ell^2}\,B_2\,B_2^*\right)X_\infty + C_1^*\,C_1 \tag{17a}$$

with initial conditions, due to (12b) and (16), given by

$$X_\infty = 0 \qquad\qquad \text{at } t = T. \tag{17b}$$

Combining (14) and (16), a robust control Φ which is effective even in the presence of a small component of the worst case disturbance

$$\boxed{\text{MLWCD:} \qquad \chi = (1/\gamma^2)\,B_1^*\,X_\infty\,\mathbf{u}}$$

is given by the state feedback relationship

$$\boxed{\Phi = K_\infty\,\mathbf{u} \qquad \text{where} \qquad K_\infty = -\frac{1}{\ell^2}\,B_2^*\,X_\infty} \tag{18}$$

where X_∞ is the solution of the time dependent Riccati equation (17) and thus $K_\infty = K_\infty(t)$.

3.6 Infinite time horizon for time invariant problems

If the matrices A, B_1, B_2, and C_1 are time invariant, then in the limit that the optimization interval $T \to \infty$ the matrix $X_\infty(t)$ defined above approaches a steady state value in the march from the initial conditions defined at $t = T$ back towards $t = 0$, and is given by the solution to

$$\boxed{0 = A^*\,X_\infty + X_\infty\,A + X_\infty\left(\frac{1}{\gamma^2}\,B_1\,B_1^* - \frac{1}{\ell^2}\,B_2\,B_2^*\right)X_\infty + C_1^*\,C_1} \tag{19}$$

The robust feedback relationship given by (18) in this limit is thus time invariant and a function of the solution to the algebraic Riccati equation (19).

4 Optimal nonlinear regulation

4.1 State equation

Consider a state vector $\mathbf{u}$ which is a function of some feedback control vector Φ such that it obeys the nonlinear evolution equation

$$\boxed{\dot{\mathbf{u}} = \mathcal{A}(\mathbf{u}) + \mathcal{B}_2(\Phi)} \tag{20a}$$

with given initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \text{at } t = 0. \tag{20b}$$

The nonlinear functions $\mathcal{A}(\mathbf{u})$ and $\mathcal{B}_2(\Phi)$ may themselves be functions of time.

4.2 Cost function

The object of applying control in the present case is identical to the optimal linear regulation problem described in §2.2. It is expressed as the minimization of

$$\boxed{\mathcal{J}_2 \equiv \frac{1}{2\,T}\int_0^T \left(\mathbf{u}^*\, C_1^*\, C_1\, \mathbf{u} + \ell^2\, \Phi^*\, \Phi\right)\, dt} \tag{21}$$

A technique to determine the control Φ on the interval $(0, T]$ which (locally) minimizes the cost function $\mathcal{J}_2$ for the nonlinear state equation (20) is now briefly outlined.

4.3 Perturbation equation

Consider the linear problem of a small perturbation $(\Phi', \mathbf{u}')$ to some reference solution $(\Phi, \mathbf{u})$ of the system given by (20). It is easily shown that such a perturbation must obey a linear evolution equation of the form

$$\boxed{\dot{\mathbf{u}}' = A\,\mathbf{u}' + B_2\,\Phi'} \tag{22a}$$

with initial conditions

$$\mathbf{u}' = 0 \qquad \text{at } t = 0. \tag{22b}$$

The matrices A and B_2 are functions of time and depend explicitly on the reference condition $(\Phi, \mathbf{u})$.

4.4 Adjoint equation

Define an adjoint state based on the A matrix of the perturbation problem (22) such that

$$-\dot{\lambda} = A^*\,\lambda + C_1^*\, C_1\, \mathbf{u} \tag{23a}$$

with initial conditions

$$\lambda = 0 \qquad \text{at } t = T. \tag{23b}$$

4.5 Gradient of cost with respect to control

As in the linear case, the gradient of the cost function $\mathcal{J}_2$ with respect to the control Φ is a simple function of the adjoint state defined by (23):

$$\frac{\mathscr{D}\mathcal{J}_2}{\mathscr{D}\Phi} = B_2^*\,\lambda + \ell^2\,\Phi. \tag{24}$$

4.6 Solution of control problem

The most suitable control on $(0, T]$ which results in

$$\frac{\mathscr{D}\mathcal{J}_2}{\mathscr{D}\Phi} = 0 \qquad \text{(minimum)} \tag{25}$$

may *not*, strictly speaking, be found simply by setting the gradient $\mathscr{D}\mathcal{J}_2/\mathscr{D}\Phi$ in (24) equal to zero, as this gradient information is accurate only in a small neighborhood of the reference solution upon which the matrices A and B_2 were based.[1] Instead, a more stable iterative approach is used based on the gradient vector:

$$\Phi^k = \Phi^{k-1} - \alpha\frac{\mathscr{D}\mathcal{J}_2}{\mathscr{D}\Phi}, \tag{26}$$

where k indicates the iteration index and α is a (positive) descent parameter to be chosen. The condition (25) is approached iteratively according to the following procedure:

1. Initialize control Φ on $(0, T]$ to $\Phi = 0$.
2. Determine state $\mathbf{u}$ on $(0, T]$ from state equation (20).
3. Determine adjoint λ on $[0, T)$ from adjoint equation (23).
4. Determine local expression for gradient $\mathscr{D}\mathcal{J}_2/\mathscr{D}\Phi$ from (24).
5. Test various different values for the scalar α in (26), computing the resulting state $\mathbf{u}$ from (20) and the resulting cost $\mathcal{J}_2$ from (21), and determine via a line minimization algorithm that value of α which results in the smallest $\mathcal{J}_2$.
6. Update control Φ on $(0, T]$ via (26) using best value of α determined in step 5.
7. Repeat from step 6 until convergence.

[1] One may propose a Newton-Raphson technique to determine the control, setting the local expression for $\mathscr{D}\mathcal{J}_2/\mathscr{D}\Phi$ in (24) equal to zero to determine a new control, determining a new reference state from (20), solving the new adjoint problem to determine a new value for Φ by again setting $\mathscr{D}\mathcal{J}_2/\mathscr{D}\Phi$ in (24) equal to zero, and iterating until convergence. However, there is no way to insure that the initial guess for Φ is sufficiently close to a minimum to guarantee convergence of this approach. Iterative approaches are *essential* to guaranteeing convergence to a solution of the nonlinear problem. Note also that the solution found is not necessarily the global solution.

5 Robust nonlinear regulation

5.1 State equation

Consider the nonlinear state equation (20) with additional forcing due to an external disturbance χ

$$\boxed{\dot{\mathbf{u}} = \mathcal{A}(\mathbf{u}) + \mathcal{B}_1(\chi) + \mathcal{B}_2(\Phi)} \tag{27a}$$

with given initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \text{at } t = 0. \tag{27b}$$

The nonlinear function $\mathcal{B}_1(\chi)$ may itself be a function of time.

5.2 Cost function

The object of applying control in the present case is identical to the robust linear regulation problem described in §3.2. Mathematically, it is expressed as the minimization of a cost function $\mathcal{J}_\infty$ with respect to the control Φ while simultaneously maximizing $\mathcal{J}_\infty$ with respect to the disturbance χ, where

$$\boxed{\mathcal{J}_\infty \equiv \frac{1}{2T}\int_0^T \left(\mathbf{u}^*\, C_1^*\, C_1\, \mathbf{u} + \ell^2\, \Phi^*\, \Phi - \gamma^2 \chi^*\, \chi\right)\, dt}$$

A technique to determine the control Φ on the interval $(0, T]$ which (locally) minimizes the cost function $\mathcal{J}_\infty$ in the presence of a small component of the worst external disturbance χ forcing the state equation (27) is now briefly outlined.

5.3 Perturbation equation

Consider the linear problem of a small perturbation $(\Phi', \chi', \mathbf{u}')$ to some reference solution $(\Phi, \chi, \mathbf{u})$ of the system given by (27). It is easily shown that such a perturbation must obey a linear evolution equation of the form

$$\boxed{\dot{\mathbf{u}}' = A\,\mathbf{u}' + B_1\,\chi' + B_2\,\Phi'} \tag{28a}$$

with initial conditions

$$\mathbf{u}' = 0 \qquad \text{at } t = 0. \tag{28b}$$

The matrices A, B_1, and B_2 are functions of time and depend explicitly on the reference condition $(\Phi, \chi, \mathbf{u})$.

5.4 Adjoint equation

Define an adjoint state based on the A matrix of the perturbation problem (28) such that

$$-\dot{\lambda} = A^*\,\lambda + C_1^*\, C_1\,\mathbf{u} \tag{29a}$$

with initial conditions

$$\lambda = 0 \qquad \text{at } t = T. \tag{29b}$$

5.5 Gradients of cost w.r.t. control and disturbance

As in the linear case, the gradients of the cost function $\mathcal{J}_\infty$ with respect to the control Φ and the disturbance χ are simple functions of the adjoint state defined by (29):

$$\frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\Phi} = B_2^*\,\lambda + \ell^2\,\Phi \quad\text{and}\quad \frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\chi} = B_1^*\,\lambda - \gamma^2\,\chi. \tag{30}$$

5.6 Solution of control problem

The most suitable control and the MLWCD which result in

$$\frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\Phi} = 0 \;\;\text{(minimum)}\quad\text{and}\quad \frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\chi} = 0 \;\;\text{(maximum)}$$

may *not*, strictly speaking, be found simply by setting the gradients $\mathscr{D}\mathcal{J}_\infty/\mathscr{D}\Phi$ and $\mathscr{D}\mathcal{J}_\infty/\mathscr{D}\chi$ in (30) equal to zero, as this gradient information is accurate only in a small neighborhood of the reference solution upon which the matrices A, B_1, and B_2 were based. Instead, an iterative approach is used based on the gradient vectors:

$$\Phi^k = \Phi^{k-1} - \alpha\frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\Phi} \quad\text{and}\quad \chi^k = \chi^{k-1} + \beta\frac{\mathscr{D}\mathcal{J}_\infty}{\mathscr{D}\chi} \tag{31}$$

where k indicates the iteration index. The iteration procedure followed is analogous to that described in §4.6; in the present case, a value of α is chosen to reduce $\mathcal{J}_\infty$ while simultaneously a value of β is chosen to increase $\mathcal{J}_\infty$, and the solution marched towards values of Φ and χ which meet the min/max criteria sought.

5.7 Approximate solution for very large systems

The min/max problem described by (31) is infeasible when the state equation (27) is a model of turbulent channel flow, as the state $\mathbf{u}$ upon which the disturbance acts in this case, and therefore any general representation of the disturbance χ itself, has a very large dimension ($\mathcal{O}(10^7)$ at $Re_\tau = 180$). Thus, instead of forcing the state equation with a disturbance χ determined by the iterative approach given in (31), which is guaranteed to be stable but would present excessive computational storage requirements, we settle on a simpler, though possibly unstable, approach for the determination of χ.[2]

[2]Note that we still determine the control Φ via the stable iterative approach given in (31).

The disturbance χ^k is chosen in this approach by setting $\mathscr{D}\mathcal{J}_\infty/\mathscr{D}\chi$ in (30) directly equal to zero. Note that, taking the matrix B_1 as simply the identity matrix, the disturbance determined in this fashion is proportional to the adjoint field itself

$$\chi^k = \frac{1}{\gamma^2}\,\lambda^{k-1} \tag{32}$$

For sufficiently large γ (*i.e.*, sufficiently small disturbance χ), this should be an accurate approximation of the global maximum $\mathscr{D}\mathcal{J}_\infty/\mathscr{D}\chi = 0$, and thus should result in an accurate approximation of the "worst case" disturbance. For smaller values of γ (*i.e.*, for larger disturbance χ), this approach is not guaranteed to be stable. Trial and error will indicate for what values of γ this approach converges.

6 Conclusions

Optimal and robust control theory for discrete linear systems were first briefly reviewed from first principles. It is shown that, in the infinite time horizon limit with time-invariant system matrices, a *closed form* solution to the linear control problems may be found in terms of the solution to a Riccati equation.

Optimal control theory for discrete nonlinear system was then briefly reviewed in a similar notation. It is shown that there are many close connections to the linear problem. However, a closed form solution is *not* available, and an iterative search for the solution is required. A gradient approach, such as a simple gradient or (better) a conjugate gradient approach, to the finite-time interval optimization problem is recommended to guarantee a stable search for the optimal controls.

A new technique for robust control theory for discrete nonlinear systems is then identified by extrapolation. Such a technique seeks the best control and, simultaneously, the Murphy's Law Worst Case Disturbance (MLWCD). Such a technique holds promise for the determination of effective control algorithms for turbulent flows and other nonlinear phenomena subjected to external disturbances with control algorithms based on limited noisy observations alone.

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REFERENCES

Abergel, F. & Temam, R. 1990 On some control problems in fluid mechanics *Theor. and Comp. Fluid Dynamics* **1**, 303-325.

Bewley, T.R., Liu, S., & Agarwal, R. 1997 Optimal and robust control and estimation of linear paths to transition. Submitted to *J. Fluid Mech.*

Bewley, T.R., Moin, P., & Temam, R. 1997 Optimal control of turbulence. Under preparation for submission to *J. Fluid Mech.*

Bloch, A. 1982 *Murphy's Law and other reasons why things go wrong.* Price/Stern/Sloan.

Doyle, J.C., Glover, K., Khargonekar, P.P., & Francis, B.A. 1989 State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems *IEEE Trans. Auto. Control* **34**, 8, 831-847.

Green, M., & Limebeer, D.J.N. 1995 *Linear robust control.* Prentice-Hall.

Laub, A.J. 1991 Invariant subspace methods for the numerical solution of Riccati equations. In *The Riccati Equation* (ed. Bittaini, Laub, & Willems) 163-196. Springer.

Lewis, F.L. 1995 *Optimal Control.* Wiley.

Skogestad, S., & Postlethwaite, I. 1996 *Multivariable Feedback Control.* Wiley.

Zhou, K., Doyle, J.C., & Glover, K. 1996 *Robust and Optimal Control.* Prentice-Hall.