Renaissance Packings

symmetry & structure in a multidimensional world

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Summary

Preface

The field of n-dimensional sphere packings is elegant and mature in its mathematical development and characterization. However, it is still relatively limited in its practical applications, especially for $n > 3$. The present text intends to open up two broad new areas for profitable application of this powerful body of mathematical literature in science and engineering. Towards this end, Part I reviews the essential results available in this field (reconciling the theoretical literature for dense and rare sphere packings, which are today largely disjoint), catalogs the key properties of the principle dense and rare sphere packings and corresponding nets available up to $n = 24$ (including hundreds of values not previously known), and extends the study of regular rare sphere packings and nets to $n > 3$ dimensions (an area which up to now has been largely unexplored).

Part II then builds from the presentation in Part I to develop three new algorithms (LABDOGS, αDOGS, and latticeMADS) for performing efficient derivative-free optimization in non-differentiable problems with expensive function evaluations, leveraging the lattices derivd from dense sphere packings as an alternative to Cartesian grids to coordinate the search. We pay particular attention to the improved uniformity and nearest-neighbor configuration of the lattices used over their Cartesian alternatives, and the improvements in

efficiency of optimization algorithms coordinated by such lattices that follow as a direct consequence.

Finally, Part III builds from the presentation in Part I to develop new interconnect strategies for switchless multiprocessor computer systems, leveraging the nets derived from rare sphere packings as alternatives to Cartesian grids to establish structured, fast, and inexpensive interconnects. We pay particular attention to the improved coordination sequences facilitated by such nets over their Cartesian alternatives, and the improvements in the rate of spread of information across the computer system that follow as a direct consequence.

In the applications discussed in Parts II and III, Cartesian grids are used as the default choice today in almost all related realizations. A primary goal of this text is to subvert this dominant Cartesian paradigm and to establish, via the examples we have chosen to highlight, that significant performance gains may be realized in practical engineering applications by leveraging n-dimensional sphere packings appropriately.

A gentle introduction to sphere packing theory

An n-dimensional infinite *sphere packing* is simply an array of nodal points in $\mathbb{R}^n$ obtained via the packing of identical n-dimensional spheres. By *packing*, we mean an equilibrium configuration of spheres, each with at least 2 nearest neighbors, against which a repellant force is applied. Many packings investigated in the literature are *stable* packings, meaning that there is a restoring force associated with any small movement of any node of the packing; this requires each sphere in the (n-dimensional) packing to have at least $n+1$ neighbors. Unstable packings with lower nearest-neighbor counts are also of interest. By replacing each sphere in an n-dimensional packing with a nodal point (representing, e.g., a computer), and connecting those nodal points which are nearest neighbors, a *net* (a.k.a. *interconnect* or *contact graph*) is formed[1].

[1] As mentioned in the second-to-last paragraph of §2.3, it is natural with certain sphere packings (for example, D_n^*, A_n^r, and the packings associated with the T_n^{90} and T_n^{60} nets) to define nets which are *not* contact graphs of the corresponding sphere packings by

An n-dimensional real *lattice* (a.k.a. *lattice packing*) is a sphere packing which is shift invariant (that is, which looks identical upon shifting any nodal point to the origin); this shift invariance generally makes lattice packings simpler to describe and enumerate than their nonlattice alternatives. Note that there are many regular[2] sphere packings which are *not* shift invariant [the nonlattice packings corresponding to the honeycomb net in 2D and the diamond and quartz nets in 3D are some well-known examples]. We will focus our attention in this text on those packings and nets which are at least *uninodal* (that is, which look identical upon shifting any nodal point to the origin and rotating and reflecting appropriately). For *dense* sphere packings, from a practical perspective, lattice packings are essentially[3] as good a choice as their more cumbersome nonlattice alternatives for $n \leq 24$ in terms of the four metrics defined below (that is, for maximizing packing density and kissing number and minimizing covering thickness and quantization error). However, the best *rare* sphere packings (with small kissing number) are all nonlattice packings.

As illustrated in Table P.1 and Figure P.1, we may introduce the subject of n-dimensional sphere packings by focusing our attention first on the $n = 2$ case: specifically, on the *triangular*[4] lattice (A_2), the *square* lattice ($\mathbb{Z}^2$), and the *honeycomb* nonlattice packing (A_2^+). The characteristics of such sphere packings may be quantified by the following measures:

• The *packing radius* (a.k.a. *error-correction radius*) of a packing, ρ, is the maximal radius of the spheres in a set of identical nonoverlapping spheres centered at each nodal point.

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connecting non-nearest-neighbor points.

[2]The regularity of a nonlattice packing is quantified precisely in §**??**.

[3]For $n = 10$, 11, 13, 18, 20, and 22, there exist nonlattice packings (denoted P_{10c}, P_{11a}, P_{13a}, $\mathcal{B}_{18}^*$, $\mathcal{B}_{20}^*$, $\mathcal{A}_{22}^*$) that are 8.3%, 9.6%, 9.6%, 4.0%, 5.2%, and 15.2% denser then the corresponding best known lattice packings (Conway & Sloane 1998, p. xix); to put this into perspective, the density of Λ_{22} is over 10^6 *times* the density of $\mathbb{Z}^{22}$.

[4]Note that many in this field refer to the A_2 lattice (Figure P.1a,b) as "hexagonal". We prefer the unambiguous name "triangular" to avoid confusion with the honeycomb nonlattice packing (Figure P.1e,f).

n	**packing**	**name**	Δ	Θ	G	τ	td_{10}
2	A_2	triangular	0.9069	1.2092	0.08019	6	331
	$\mathbb{Z}^2$	square	0.7854	1.5708	0.08333	4	221
	A_2^+	honeycomb	0.6046	2.4184	0.09623	3	166
8	E_8	Gosset	0.2537	4.059	0.07168	240	1,006,201,681
	$\mathbb{Z}^8$	Cartesian	0.01585	64.94	0.08333	16	1,256,465
	(unstable)	V_8^{90}	5.590e-4	49.89	0.09206	4	37,009
		Y_8^{90}	2.327e-4	87.31	0.09266	3	2290
24	Λ_{24}	Leech	0.001930	7.904	0.06577	196560	$> 10^{15}$
	$\mathbb{Z}^{24}$	Cartesian	1.150e-10	4,200,263	0.08333	48	24,680,949,041

Table P.1. Characteristics of selected lattice and uninodal nonlattice packings and nets. Note that $n = 24$ is a natural stopping point in this study. It is special because it is the only integer $n > 1$ that satisfies the equation $1^2 + 2^2 + \ldots + n^2 = m^2$ where m is itself an integer; as a consequence, a particularly uniform lattice with a large number of symmetries is available in this dimension.

• The *packing density* of a packing, Δ, is the fraction of the volume of the domain included within a set of identical non-overlapping spheres of radius ρ centered at each nodal point on the packing. Packings that maximize this metric are referred to as *close-packed*.

• The *covering radius* of a packing, R, is the maximum distance between any point in the domain and its

nearest nodal point on the packing. The *deep holes* of a packing are those points which are at a distance R from all of their nearest neighbors. Typical vectors from a nodal point to the nearest deep holes in a lattice packing are often denoted [1], [2], etc.

• The *covering thickness* of a packing, Θ, is the number of spheres of radius R centered at each nodal point containing an arbitrary point in the domain, averaged over the domain.

• The *Voronoï cell* of a nodal point in a packing, $\Omega(P_i)$, consists of all points in the domain that are at least as close to the nodal point P_i as they are to any other nodal point P_j.

• The *mean squared quantization error per dimension* of a lattice or uninodal nonlattice packing, G, is the average mean square distance of any point in the domain to its nearest nodal point, normalized by n times the appropriate power of the volume, V, of the Voronoï cell. Shifting the origin to be at the centroid of a Voronoï cell $\Omega(P_i)$, it is given by

$$G = \frac{S}{nV^{\frac{n+2}{n}}} \quad \text{where} \quad S = \int_{\Omega(P_i)} |\mathbf{x}|^2 \, d\mathbf{x}, \quad V = \int_{\Omega(P_i)} d\mathbf{x}. \tag{1}$$

• The *kissing number* (a.k.a. *error coefficient*) of a lattice or uninodal nonlattice packing, τ, is the number of nearest neighbors to any given nodal point in the packing. That is, it is the number of spheres of radius ρ centered at the nodal points of the packing that touch, or "kiss", the sphere of radius ρ at the origin.

• The *coordination number* of a net (derived from a sphere packing, as discussed previously) is the first number of the net's *coordination sequence*, the k'th element of which is given by $td_k - td_{k-1}$, where td_k, which quantifies the net's *local topological density*, is the total number of nodes reached via k hops or less from the origin in the net[5].

[5] In most cases, the natural net to form from a sphere packing is the contact graph; in such cases, the kissing number, τ, and the

Certain applications, such as those explored in Part II, require dense lattices. There are two key drawbacks with Cartesian approaches for such applications. First, the *discretization of space is significantly less uniform* when using the Cartesian grid as opposed to the available alternatives, as measured by the packing density Δ, the covering thickness Θ, and the mean-squared quantization error per dimension, G (see Table P.1). Second, the *configuration of nearest-neighbor gridpoints is significantly more limited* when using the Cartesian grid, as measured by the kissing number τ, which is an indicator of the degree of flexibility available when selecting from nearest-neighbor points. As seen by comparing the $n = 2$, $n = 8$, and $n = 24$ cases in Table P.1, these drawbacks become increasingly substantial as the dimension n is increased; by the dimension $n = 24$, the Cartesian grid has

- a factor of $0.001930/1.1501e-10 \approx 17,000,000$ worse (lower) packing density,
- a factor of $4,200,263/7.9035 \approx 530,000$ worse (higher) covering thickness,
- a factor of $0.08333/0.0658 \approx 1.27$ worse (higher) mean-squared quantization error, and
- a factor of $196560/48 \approx 4100$ worse (lower) kissing number

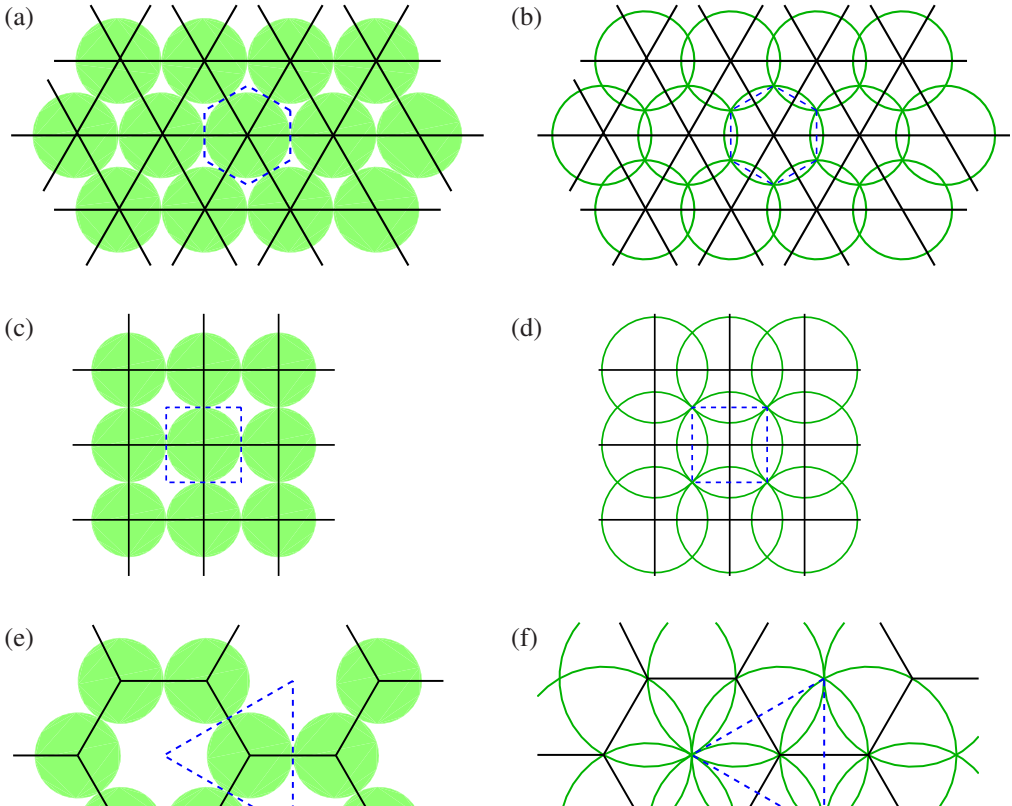
than the densest available alternative lattice. Thus, the selection of the Cartesian grid, by default, for applications requiring dense (that is, uniform) lattices with $n > 3$ is simply untenable.

Other applications, such as those explored in Part III, require regular nets which, with low coordination number, connect to a large number of nodes with each successive hop from the origin, as quantified by the net's coordination sequence. As mentioned previously, a useful measure of a net's topological density is given, e.g., by td_{10}, which is the number of distinct nodes within 10 hops of the origin. Note that the coordination

coordination number are equal. As mentioned previously, it is natural with certain sphere packings to define nets which are *not* contact graphs by connecting non-nearest-neighbor points; in such cases, the kissing number (a property of the sphere packing) and the coordination number (as defined here, a property of a corresponding net) are, in general, *not* equal. We find this clear semantical distinction to be useful to prevent confusion between these two distinct concepts; note that some authors (e.g., Conway & Sloane 1998) do not make this distinction.

number of the n-dimensional Cartesian grid is $2n$; the coordination number of the alternative n-dimensional constructions introduced in §**??** are as small as 3 or 4, while the topological density increases rapidly as n is increased (compare, e.g., the values of td_{10} for A_2^+ and $\mathbb{Z}^2$, with $\tau = 3$ and $\tau = 4$ respectively, to those for Υ_8^{90} and V_8^{90} in Table P.1); it is thus seen that, for applications requiring graphs with low coordination number and high topological density, the selection of the Cartesian grid, by default, is also untenable.

We are thus motivated to make the fundamental results of both dense and rare n-dimensional sphere packing theory more broadly accessible to the science and engineering community, and to illustrate how this powerful body of theory may be put to use in important new applications of practical relevance. Towards this end, Part I succinctly reviews and extends several significant results in this mature and sophisticated field, inter-relating the literature on dense and rare packings, which is today largely disjoint. These results are leveraged heavily in the applications described in Parts II and III. We note that, beyond providing an up-to-date and synthetic review of this otherwise difficult subject in a (hopefully) accessible language, a significant number of new computations, constructions, algorithms, metrics, and codes are also reported in Part I [the reader is referred specifically to §**??**, §**??** through §**??**, §**??**, and §**??**].


(a)
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Part I

Fundamental concepts & constructions, from dense to rare

Chapter 1

Historical retrospective

Contents

The mathematical characterization of sphere packings has a long and rich history. Some recent articles and popular books recount this history in detail, including Zong (1999), Szpiro (2003), Hales (2006), and Aste & Weaire (2008). The purpose of the present Part I is not to repeat these historical retrospectives, which these

sources do quite adequately, but to characterize, catalog, and extend the infinite packings available today to facilitate their practical application in new fields. Nonetheless, we would remiss if we didn't at least provide a brief historical context to this field, which we attempt in this short chapter.

1.1 Finite packings

Mystic marbles. We begin by defining, for $m \geq 1$, a notation to build from:

$$T_{0,m} \triangleq 1, \qquad T_{1,m} \triangleq \sum_{k=1}^{m} T_{0,k} = m \ \text{ (the positive integers).}$$

In the sixth century BC, Pythagoras and his secret society of numerologists, the Pythagoreans, discovered geometrically (see Figure 1.1, and pp. 43-50 of Heath 1931) the formula for the number of marbles placed in a (2D) triangle (that is, the "triangular numbers"):

$$T_{2,m} \triangleq \sum_{k=1}^{m} T_{1,k} = m(m+1)/2.$$

Stacked spheres. The earliest known mathematical work to discuss the (3D) stacking of objects is a Sanskrit document *The Aryabhatiya of Aryabhata* (499 AD; see Clark 1930, p. 37), which states:

"In the case of an *upaciti* [lit., 'pile'] which has ... the product of three terms, having the number of terms for the first term and one as the common difference, divided by six, is the *citighana* [lit., 'cubic contents of the pile']. Or, the cube of the number of terms plus one, minus the cube root of this cube, divided by six."

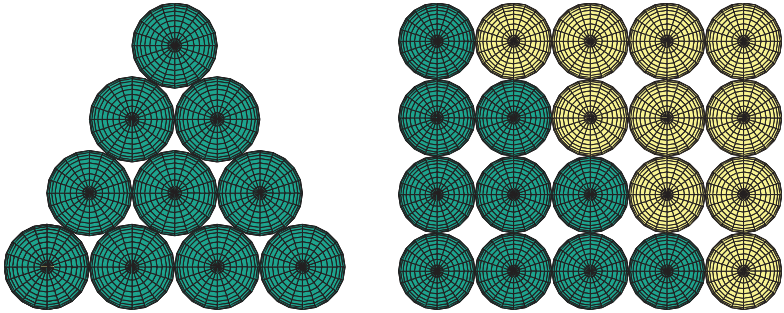

Figure 1.1: (left) Ten marbles placed in a triangle [referred to by the Pythagoreans as a τετρακτύς, and upon which they placed a particular mystic significance], and (right) the Pythagoreans' placement of two triangular groups of marbles into an "oblong" $m \times (m+1)$ rectangle, from which the formula for $T_{2,m}$ follows immediately.

Thus, Aryabhata establishes, in words, two equivalent expressions for the number of objects ("cubic contents") in a (3D) triangular-based pyramid ("pile") with m objects on each edge:

$$T_{3,m} = \frac{m(m+1)(m+2)}{3!} = \frac{(m+1)^3 - (m+1)}{6};$$

note also that $T_{3,m} \triangleq \sum_{k=1}^{m} T_{2,k}$.

Thomas Harriot was apparently the first to frame the problem of sphere packing mathematically in modern times (see, e.g., the biography of Harriot by Rukeyser 1972). At the request of Sir Walter Raleigh, for whom

Harriot served, among other capacities, as an instructor of astronomical navigational and on various problems related to gunnery, Harriot (on December 12, 1591) computed, but did not publish, the number of cannonballs in a pile with a triangular, square $[m \times m]$, and rectangular $[m \times (m+1)]$, a.k.a. "oblong"] base, as illustrated in Figure 1.2, obtaining $T_{3,m}$, S_m, and R_m respectively, where

$$S_m = \sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}, \quad R_m = \sum_{k=1}^{m} k(k+1) = S_m + T_{2,m} = \frac{m(m+1)(2m+4)}{6}.$$

In 1614, Harriot wrote *De Numeris Triangularibus Et inde De Progressionibus Artithmeticis: Magisteria magna* (*On triangular numbers and thence on arithmetic progressions: the great doctrine*)[1]. Looking closely at the triangular table of binomial coefficients[2] on pp. 1-3 (folios 108-110) of this remarkable document, it is seen that Harriot understood the *geometric* relationship between the positive integers $T_{1,m}$, the "triangular numbers" $T_{2,m}$ [that is, the number of spheres in a (2D) triangle with m spheres on each edge], the "pyramidal numbers" $T_{3,m}$ [that is, the number of spheres in a (3D) trianglar-based pyramid with m spheres on each edge], and the next logical steps in this arithmetic progression, given by:

$$T_{4,m} \triangleq \sum_{k=1}^{m} T_{3,k} = \frac{m(m+1)(m+2)(m+3)}{4!}, \quad T_{5,m} \triangleq \sum_{k=1}^{m} T_{4,k} = \frac{m(m+1)(m+2)(m+3)(m+4)}{5!},$$

etc. In particular, Harriot noticed that the $(n+1)$'th element of the $(n+m)$'th row of this triangular table is $T_{n,m}$. Accordingly, we may think of $T_{n,m}$ as the number of spheres in an "n-dimensional pyramid" with m spheres on each edge, with $T_{n,2}$ representing $n+1$ spheres configured at the corners of a regular n-dimensional

[1] Harriot (1614) passed through several hands before finally being published in 2009, almost 4 centuries later.

[2] This famous triangular table of binomial coefficients is incorrectly attributed by many in the west to Blaise Pascal (b. 1623), though it dates back to several earlier sources, the earliest being Pingala's Sanskrit work *Chandas Shastra*, written in the fifth century BC.

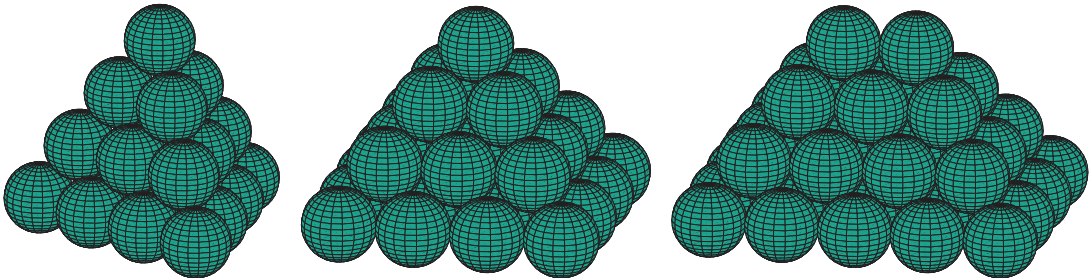

Figure 1.2: Pyramidal stacks of spheres with triangular, square, and "oblong" (rectangular) bases. All three stacks are subsets of the face-centered cubic lattice, discussed further in §2.3.

simplex. It is thus natural to credit Harriot (1614) with the first important steps towards the discovery of laminated lattices, discussed further in §2.4 and §2.6.

Harriot also introduced the packing problem to Johannes Kepler, ultimately leading Kepler (1611), in another remarkable document *Strena seu de nive sexangula* (*The six-cornered snowflake*), which also hypothesized about a related atomistic physical basis for hexagonal symmetry in crystal structures of water, to conjecture that

"The (cubic or hexagonal close) packing is the tightest possible, such that in no other arrangement can more spheres be packed into the same container."

Kepler's conjecture is patently false if considered in a finite container of a specified shape. For instance, a $2d \times 2d \times 2d$ cubic container can fit 8 spheres of diameter d if arranged in Cartesian configuration, but can

only fit 5 spheres if arranged in a "close-packed" configuration[3]. It is presumed that Kepler in fact recognized this, and thus Kepler's conjecture is commonly understood as a conjecture regarding the densest packing possible in the limit that the size of the container is taken to infinity (for further discusssion, see §1.2).

Permuted planets. Note in Figure 1.2 that any sphere (referred to as a "sun") on the interior of the piles has 12 nearest neighbors (referred to as its "planets"). Considering this sun and its 12 planets in isolation, there is in fact adequate room to permute the planets to different positions while keeping them in contact with the sun, something like a 12-cornered Rubik's cube with spherical pieces (see Figure 1.3). Due to the extra space available in this configuration, it is unclear upon first inspection whether or not there is sufficient room to fit a 13'th planet in to touch the sun while keeping all of the other 12 planets in contact with it. In 1694, Isaac Newton conjectured this could not be done, in a famous disagreement with David Gregory, who thought it could. Newton turned out to be right, with a complete proof first given in Schütte & van der Waerden (1953), and a substantially simplified proof given in Leech (1956).

Cartoned cans. Moving from 16th-century stacks of cannonballs to 21st-century commerce, the question of dense finite packings of circles and spheres finds practical relevance in a variety of packaging problems. For example, to form a rectangular cardboard carton for 12 fl oz soda cans, 164 cm^2 of cardboard per can is needed if 18 cans are placed in a cartesian configuration with 3 rows of 6 cans per row, whereas 3.3% less cardboard per can is needed if 18 cans are placed in a triangular configuration (within a rectangular box) with 5 rows of {4,3,4,3,4} cans per row. If an eye-catching (stackable, strong, "green"...) hexagonal cardboard carton for the soda cans is used, with 19 cans (described in marketing terms as "18 plus 1 free") again placed in a triangular configuration, 17.7% less cardboard per can is required.

[3]For larger containers, the arrangements which pack in the greatest number of spheres (or other objects) must in general be found numerically (see Gensane 2004, Schürmann 2006, and Friedman 2009).

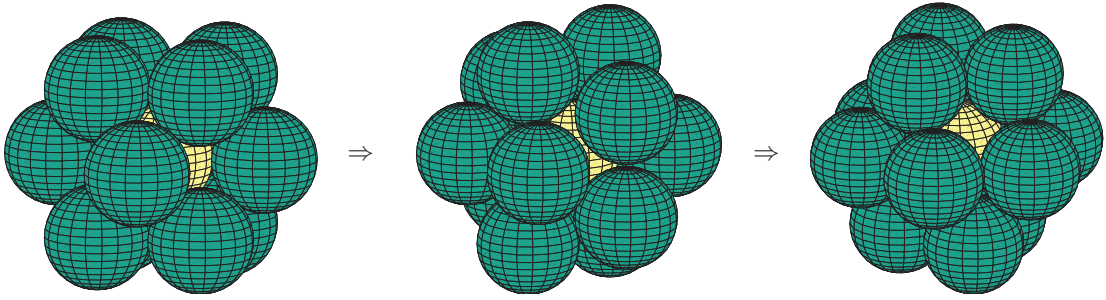

Figure 1.3: Illustration of the 13 spheres (a.k.a. Newton-Gregory) problem and planetary permutations. Configuration (a) is 13 of the spheres taken from the second, third, and fourth layers of the stack in the orientation shown in Figure 1.2b, whereas configuration (c) is 13 of the spheres taken from the third, fourth, and fifth layers of the stack in the orientation shown in Figure 1.2a [extended by one additional layer]. In both configurations, the 12 "planets" (positioned around the central "sun") are centered at the vertices of a cuboctahedron. The planets can be permuted by "pinching" together two of the four planets on the corners of each square face, in an alternating fashion, to form a symmetric icosahedral configuration with significant space between each pair of planets [configuration (b)], then "pushing" apart pairs of planets in an analogous fashion to form a different cuboctahedron. Alternatively, starting from configuration (b), identifying any pair of opposite planets as "poles", and slightly shifting the five planets in each of the "tropics" as close as possible to their nearest respective poles, the resulting northern and southern groupings of planets can be rotated in relation to each other along the equator. Repeated application of these two fundamental motions can be used to permute the planets arbitrarily.

Catastrophic sausages. Two new questions arise when one "shrink-wraps" a number (m) of n-dimensional spheres (resulting in a convex, fitted container), namely: what configuration of the spheres minimizes the surface area of the resulting container, and what configuration minimizes the volume of the resulting container? Both questions remain open, and are reviewed in Zong (1999). Regarding the minimim surface area question, it was conjectured by Croft, Falconer, & Guy (1991) that the minimum surface area, for $n \geq 2$ and large m, is achieved with a roughly spherical arrangement. In contrast, regarding the minimim volume question, it was conjectured by L. Fejes Tóth (1975) that the minimum volume, for $n \geq 5$ and any m, is achieved by placing the spheres in a line, leading to a shrink-wrapped container in the shape of a "sausage". For $n = 3$, it has been shown that a roughly spherical arrangement minimizes the volume for $m = 56$, $m = 59$ to 62, and $m \geq 65$, and it is conjectured that a sausage configuration minimizes the volume for all other m (see Gandini & Willis 1992); for $n = 4$, there appears to be a similar "catastrophe" in the volume-minimizing solution, from a sausage configuration to a roughly spherical configuration, as m is increased beyond a critical value (Willis 1983 conjectures this critical value to be $m \approx 75000$, whereas Gandini & Zucco 1992 conjectures it to be $m = 375769$).

Concealed origins. Finally, L. Fejes Tóth (1959) presents a curious set of questions that arise when considering the blocking of light with a finite number of opaque unit spheres packed around the origin. The first such question, known as Hornich's Problem, seeks the smallest number of opaque unit spheres that completely conceal light rays emanating from a point source at the center of a transparent unit sphere at the origin. A related question, known as L. Fejes Tóth's Problem, seeks the smallest number of opaque spheres that completely conceal light rays emanating from the surface of a unit sphere at the origin (e.g., in Figure 1.3, adding additional outer planets to completely conceal the view of the sun from all angles). In 2D, the (trivial) answer to both problems is 6, via the triangular packing indicated in Figure P.1a. In higher dimensions, both questions remain open, and the answer differs depending on whether or not the sphere centers are restricted to

the nodal points of a lattice. For the L. Fejes Tóth's Problem, for $n \geq 3$, the answer is unbounded if restricted to lattice points, and bounded if not. For Hornich's Problem, the answer is bounded in both cases, with the number of spheres, h, required in the 3D case, when not restricted to lattice points, being somewhere in the range $30 \leq h \leq 42$. Zong (1999) derives several of the known bounds available in both problems.

1.2 Infinite packings

In the last 300 years, *many* different constructions of infinite lattice and nonlattice packings have been proposed in each dimension. These packings each have different packing density, covering thickness, mean-squared quantization error, and kissing number, and their corresponding nets each have different topological density; knowledge of these properties is essential when selecting a packing or net for any given application. We have thus attempted to catalog these constructions and their properties thoroughly in this review (see §**??**).

In the characterization of density, amongst all *lattice* packings of a given dimension, the A_2, A_3, D_4, D_5, E_6, E_7, E_8, and Λ_{24} constructions given in §2 have been proven to be of maximum density, in Lagrange (1773) for $n = 2$, Gauss (1831) for $n = 3$, Korkine & Zolotareff (1873, 1877) for $n = 4$ and 5, Blichfeldt (1935) for $n = 6$ through 8, and Cohn & Kumar (2009) for $n = 24$. There are no such proofs of optimality for other values of n, though the lattices Λ_n and K_n introduced in §2.6 are likely candidates in the range $9 \leq n \leq 23$.

Remarkably, if one considers both lattice *and* nonlattice packings, proof of which packing is of maximum density in a given dimension is still open for $n > 3$. It was established in Thue (1892) that A_2 has the maximum density amongst all lattice and nonlattice packings for $n = 2$. Considerable attention has been focused over the centuries on the corresponding question for A_3 in dimension $n = 3$, that is, on Kepler's conjecture (posed in 1611) in the limit that the container size is taken to infinity. Indeed, David Hilbert, in his celebrated list

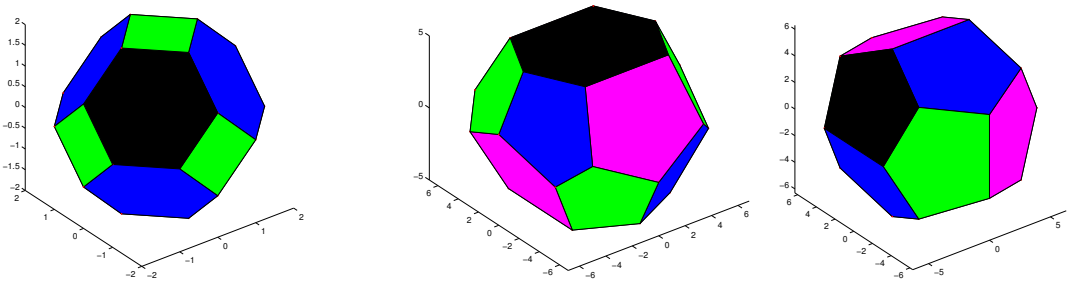

Figure 1.4: (a) A regular truncated octahedron, used to tile $\mathbb{R}^3$ in Kelvin's conjecture; (b) an irregular tetrakaidecahedron and dodecahedron, used to tile $\mathbb{R}^3$ in the Weaire-Phelan structure.

of 23 significant open problems in mathematics in 1900, included a generalization of Kepler's conjecture as part of his 18th problem (see, e.g., Milnor 1976).

Note that it is not at all obvious that an infinite packing as regular as A_3 would necessarily be the packing that maximizes density. Indeed, as mentioned in footnote 3 on page vii, nonlattice packings are known in dimensions $n = 10, 11, 13, 18, 20$, and 22 that are each slightly denser than the densest known lattice packings in these dimensions.

In three dimensions, physiologist Stephen Hales (1727), in his groundbreaking work *Vegetable Staticks*, reported a curious experiment:

"I compressed several fresh parcels of Pease in the same Pot, ... by the great incumbent of weight, pressed into the interstices of the Pease, which they adequately filled up, being therefore formed into pretty regular dodecahedrons."

This report implied that many of the dilated peas in this experiment had 12 nearest neighbors and/or pentagonal faces. However, the "pretty regular" qualification left a certain ambiguity, and this experiment left mathematicians puzzled, as it is patently impossible to tile $\mathbb{R}^3$ with regular dodecahedra. Kelvin (1887) formalized the question inherent in Hales' dilated pea experiment by asking how $\mathbb{R}^3$ could be divided into regions of equal volume while minimizing the partitional area. He conjectured the answer to be a regular tiling of $\mathbb{R}^3$ with truncated octahedra, which are in fact the Voronoï cells of the A_3^* lattice (see §**??**). [Note that the Voronoï cell of the A_3 lattice is the (face-transitive) *rhombic* dodecahedron, which is dual to the cuboctahedron illustrated in Figures 1.3a,c and tiles $\mathbb{R}^3$ with slightly greater partitional area than does the tiling with truncated octahedra.] Kelvin's conjecture stood for over 100 years, until Weaire & Phelan (1994) discovered a tiling of $\mathbb{R}^3$ based on irregular tetrakaidecahedra (with 2 hexagonal faces and 12 pentagonal faces) and irregular dodecahedra (with 12 pentagonal faces); this tiling has 0.3% less partitional area than the much more regular tiling with truncated octahedra considered by Kelvin (see Figure 1.4). In hindsight, it is quite possible that Hales might have in fact stumbled upon the Weaire-Phelan structure in his cooking pot (in 1727!) and, seeing all of those pentagonal faces and 12-sided (as well as 14-sided) dilated peas, asserted that what he was looking at was a culinary approximation to a tiling of $\mathbb{R}^3$ with regular dodecahedra, even though such a tiling is impossible.

Returning to Kepler's conjecture, in 1998, Thomas Hales (no relation to Stephen) announced a long-sought-after proof, in a remarkably difficult analysis making extensive use of computer calculations. This proof was spread over a sequence of papers published in the years that followed (see Hales 2005). An extensive discussion of this proof, which is still under mathematical scrutiny, is given in Szpiro (2003). Inspiration for this proof was based, in part, on a strategy to prove Kepler's conjecture proposed by L. Fejes Tóth (1953), the first step of which is a quantitative version of the Newton-Gregory problem discussed in §1.1.

Chapter 2

Dense lattice packings for $n \leq 24$

Contents

There are many dense lattices more complex than the Cartesian lattice that offer superior uniformity and nearest-neighbor configuration, as quantified by the standard metrics introduced in the Preface (namely, packing density, covering thickness, mean-square quantization error, and kissing number). This section provides an overview of many of these lattices; the definitive comprehensive reference for this subject is Conway & Sloane (1998), to which the reader is referred for much more detailed discussion and further references on many of the topics discussed in this chapter. The subject of coding theory, reviewed in §**??**, is closely related to the subject of such dense lattice packings (see also §**??**). As mentioned in the Preface, the practical applications explored in Part II of this text leverage these constructions heavily.

2.1 Lattice terminology

The notation $L_n \cong M_n$ means that the lattices L_n and M_n are *equivalent* (when appropriately rotated and scaled) at the specified dimension n. Also note that the four most basic families of lattices introduced in this chapter, denoted $\mathbb{Z}^n$, A_n, D_n, and E_n, are often referred to as *root lattices* due to their relation to the root systems of Lie algebra.

There are three primary methods[1] to define any given n-dimensional real lattice:

[1] A convenient alternative method for building a cloud of lattice points near the origin is based on the stencil of nearest-neighbor points to the origin in the lattice, repeatedly shifting this stencil to each of the lattice points near the origin determined thus far in order to create additional lattice points in the cloud. Unfortunately, this simple alternative method does not work for all lattices, such as D_n^* and A_n^r (see §2.3 and 2.4).

- As an *explicit description* of the points included in the lattice.
- As an *integer linear combination* (that is, a linear combination with integer coefficients) of a set of n basis vectors $\mathbf{b}^i$ defined in $\mathbb{R}^{n+m}$ for $m \geq 0$; for convenience, we arrange these basis vectors as the columns[2] of a *basis matrix*[3] B.
- As a *union of cosets*, or sets of nodal points, which themselves may or may not be lattices.

The standard forms of these definitions, as used throughout this chapter, make it straightforward to generalize application codes that can build easily upon any of the lattices so described.

Any real (or complex) lattice L_n has associated with it a *dual lattice* L_n^* defined such that

$$L_n^* = \left\{ \mathbf{x} \in \mathbb{R}^n \text{ (or } \mathbb{C}^n) : \mathbf{x} \cdot \bar{\mathbf{u}} \in \mathbb{Z} \text{ for all } \mathbf{u} \in L_n \right\}, \tag{2.1}$$

where $\mathbb{Z}$ denotes the set of all integers, dot denotes the usual scalar product, and overbar denotes the usual complex conjugate. If B is a square basis matrix for L_n, then B^{-T} is a square basis matrix for L_n^*.

Unless specified otherwise, the word lattice in this paper implies a real lattice, defined in $\mathbb{R}^n$. However, note that it is straightforward to extend this work to complex lattices, defined in $\mathbb{C}^n$. To accomplish this extension, it is necessary to extend the concept of the integers, which are used to construct a lattice via the "integer" linear combination of the basis vectors in a basis matrix B, as described above. There are two primary such extensions:

[2] In the literature on this subject, it is more common to use a *generator matrix* M to describe the construction of lattices. The basis matrix convention B used here is related simply to the corresponding generator matrix such that $B = M^T$; we find the basis matrix convention to be more natural in terms of its linear algebraic interpretation.

[3] Note that integer linear combinations of the columns of most matrices do *not* produce lattices (as defined in the second paragraph of the "gentle introduction" of the Preface). The matrices listed in §2 as basis matrices are special in this regard. Note also that basis matrices are not at all unique, but the lattices constructed from alternative forms of them are equivalent; the forms of the basis matrices listed in §2 were selected based on their simplicity.

- The *Gaussian integers*, defined as $\mathscr{G} = \{a+b\mathrm{i} \,:\, a,b \in \mathbb{Z}\}$ where $\mathrm{i} = \sqrt{-1}$, which lie on a square array in the complex plane $\mathbb{C}$.
- The *Eisenstein integers*, defined as $\mathscr{E} = \{a+b\omega \,:\, a,b \in \mathbb{Z}\}$ where $\omega = (-1+\mathrm{i}\sqrt{3})/2$ [note that $\omega^3 = 1$], which lie on a triangular array in the complex plane $\mathbb{C}$.

We may thus define three types of lattices from a basis matrix B:

- a real lattice, defined as a linear combination of the columns of B with integers as weights;
- a (complex) $\mathscr{G}$ lattice, defined as a linear combination of the columns of B with Gaussian integers as weights; and
- a (complex) $\mathscr{E}$ lattice, defined as a linear combination of the columns of B with Eisenstein integers as weights.

The special n-dimensional real, $\mathscr{G}$, and $\mathscr{E}$ lattices formed by taking $B = I_{n\times n}$ are denoted $\mathbb{Z}^n$, $\mathbb{Z}[\mathrm{i}]^n$, and $\mathbb{Z}[\omega]^n$ respectively. Note also that, for any complex lattice with elements $\tilde{\mathbf{z}} \in \mathbb{C}^n$, there is a corresponding real lattice with elements $\tilde{\mathbf{x}} \in \mathbb{R}^{2n}$ such that

$$\tilde{\mathbf{x}} = \begin{pmatrix} \Re\{\tilde{z}_1\} & \Im\{\tilde{z}_1\} & \ldots & \Re\{\tilde{z}_n\} & \Im\{\tilde{z}_n\} \end{pmatrix}^T. \tag{2.2}$$

The present sequence of papers focuses on the practical use of real lattice and nonlattice packings with $n > 3$. Thus, in the present Part I, we only make brief use of complex lattices to simplify certain constructions.

2.2 The Cartesian lattice $\mathbb{Z}^n$

The *Cartesian lattice*, $\mathbb{Z}^n$, is defined $\mathbb{Z}^n = \{(x_1,\ldots,x_n) : x_i \in \mathbb{Z}\}$, and is constructed via integer linear combination of the columns of the basis matrix $B = I_{n\times n}$. The Cartesian lattice is self dual $[(\mathbb{Z}^n)^* \cong \mathbb{Z}^n]$.

2.3 The checkerboard lattice D_n and its dual D_n^*

The *checkerboard lattice*, D_n, is an n-dimensional extension of the 3-dimensional *face-centered cubic* (*FCC*, a.k.a. *cubic close packed*) lattice. It is defined

$$D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \ldots + x_n = \text{even}\}, \tag{2.3a}$$

and may be constructed via integer linear combination of the columns of the $n \times n$ basis matrix

$$B_{D_n} = \begin{pmatrix} -1 & 1 & & & 0 \\ -1 & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ 0 & & & & -1 \end{pmatrix}. \tag{2.3b}$$

The dual of the checkerboard lattice, denoted D_n^* and reasonably identified as the *offset Cartesian lattice*, is an n-dimensional extension of the 3-dimensional *body-centered cubic* (*BCC*) lattice. It may be written as

$$D_n^* = D_n \cup ([1] + D_n) \cup ([2] + D_n) \cup ([3] + D_n) \cong \mathbb{Z}^n \cup ([1] + \mathbb{Z}^n), \tag{2.4a}$$

where the *coset representatives* $[1]$, $[2]$, and $[3]$ are defined in this case such that

$$[1] = \begin{pmatrix} 1/2 \\ \vdots \\ 1/2 \\ 1/2 \end{pmatrix}, \quad [2] = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad [3] = \begin{pmatrix} 1/2 \\ \vdots \\ 1/2 \\ -1/2 \end{pmatrix}.$$

The D_n^* lattice may also be constructed via integer linear combination of the columns of the $n \times n$ basis matrix

$$B_{D_n^*} = \begin{pmatrix} 1 & & & 0 & 0.5 \\ & 1 & & & 0.5 \\ & & \ddots & & \vdots \\ & & & 1 & 0.5 \\ 0 & & & & 0.5 \end{pmatrix}. \tag{2.4b}$$

It is important to recognize that, for $n \geq 5$, the contact graph of the D_n^* lattice is simply two disjoint nets given by the contact graphs of the $\mathbb{Z}^n$ and shifted $\mathbb{Z}^n$ sets of lattice points upon which D_n^* may be built [see (2.4a)]. Thus, as suggested by Conway & Sloane (1997), we introduce, for $n \geq 4$, a *generalized net* formed by connecting each node of the unshifted $\mathbb{Z}^n$ set to the 2^n nearest nodes on the shifted $\mathbb{Z}^n$ set, and each node on the shifted $\mathbb{Z}^n$ set to the 2^n nearest nodes on the unshifted $\mathbb{Z}^n$ set. The resulting net, of coordination number 2^n, is uninodal, but is *not* a contact graph of the corresponding sphere packing.

2.3.1 The offset checkerboard packing D_n^+

The packing D_n^+, reasonably identified as the *offset checkerboard packing*, is an n-dimensional extension of the 3-dimensional *diamond* packing, and is defined simply as

$$D_n^+ = D_n \cup ([1] + D_n); \tag{2.5}$$

note that D_n^+ is a lattice packing only for even n, and that D_3^+ is the *diamond packing* (for further discussion, see §**??**).

2.4 The zero-sum lattice A_n and its dual A_n^*

The *zero-sum lattice*, A_n, may be thought of as an n-dimensional extension of the 2-dimensional *triangular lattice*; in 3 dimensions, $A_3 \cong D_3$. It is defined

$$A_n = \{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} : x_0 + \ldots + x_n = 0\}, \tag{2.6a}$$

and may be constructed via integer linear combination of the columns of the $(n+1) \times n$ basis matrix

$$B_{A_n} = \begin{pmatrix} -1 & & & 0 \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ 0 & & & 1 \end{pmatrix}, \qquad \text{with} \qquad \mathbf{n}_{A_n} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}. \tag{2.6b}$$

Notice that A_n is constructed here via n basis vectors in $n+1$ dimensions. The resulting lattice lies in an n-dimensional subspace in $\mathbb{R}^{n+1}$; this subspace is normal to the vector $\mathbf{n}_{A_n}$. An illustrative example is A_2, the triangular $2D$ lattice, which may conveniently be constructed on a plane in $\mathbb{R}^3$ (see Figure 2.1).

Note that, starting from a (2D) triangular configuration of oranges or cannonballs (see Figure P.1a), one can stack additional layers of oranges in a trangular configuration on top, appropriately offset from the base layer, to build up the (3D) FCC configuration mentioned previously (see Figure 1.2a). This idea is referred to as *lamination*, and will be extended further in §2.6 when considering the Λ_n and K_n families of lattices.

Also note that, in the special case of $n = 2$, the A_2 lattice may also be written as

$$A_2 \cong R_2 \cup (\mathbf{a} + R_2), \qquad \text{where} \quad \mathbf{a} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \tag{2.6c}$$

and R_2 is the *rectangular grid* (not a lattice, nor even a nonlattice packing) obtained by stretching the $\mathbb{Z}^2$ lattice in the second element by a factor of $\sqrt{3}$.

The dual of the zero-sum lattice, denoted A_n^*, may be written as

$$A_n^* = \bigcup_{s=0}^{n} ([s] + A_n), \tag{2.7a}$$

where the $n+1$ coset representatives $[s]$, for $s = 0, \ldots, n$, are defined such that the k'th component of the vector $[s]$ is

$$[s]_k = \begin{cases} \frac{s}{n+1} & k \leq n+1-s, \\ \frac{s-n-1}{n+1} & \text{otherwise.} \end{cases} \tag{2.7b}$$

The A_n^* lattice may be constructed via integer linear combination of the columns of the $(n+1) \times n$ basis matrix

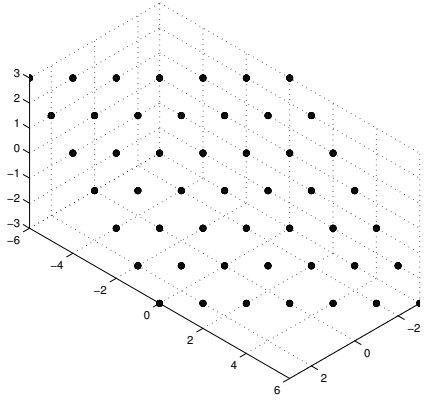

Figure 2.1: A cloud of points on the A_2 lattice, defined on a plane in $\mathbb{R}^3$. Note that the normal vector $\mathbf{n}_{A_2} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ points directly out of the page in this view.

$$B_{A_n^*} = \begin{pmatrix} 1 & 1 & \cdots & 1 & \frac{-n}{n+1} \\ -1 & & & 0 & \frac{1}{n+1} \\ & -1 & & & \frac{1}{n+1} \\ & & \ddots & & \vdots \\ & & & -1 & \frac{1}{n+1} \\ 0 & & & & \frac{1}{n+1} \end{pmatrix}, \qquad \text{with} \qquad \mathbf{n}_{A_n^*} = \mathbf{n}_{A_n}. \tag{2.7c}$$

2.4.1 The glued zero-sum lattices A_n^r

A related family of lattice packings, developed in §12 of Coxeter (1951) and reasonably identified as the *glued zero-sum lattices* A_n^r, is a family of lattices somewhere between A_n and A_n^* [as given in (2.7a)] defined via the union of r translates of A_n for $n \geq 5$:

$$A_n^r = A_n \cup ([s] + A_n) \cup ([2s] + A_n) \cup \ldots \cup ([(r-1)s] + A_n), \quad \text{where} \quad r \cdot s = n+1, \tag{2.8}$$

where the components of the "glue" vectors $[s]$ are specified in (2.7b), and where r and s are integer divisors of $(n+1)$ with $1 < s < n+1$ and $1 < r < n+1$, excluding the case $\{r = 2, s = 3\}$ for $n = 5$. The lattices A_9^5, A_{11}^4, A_{13}^7, A_{14}^5, A_{15}^8, A_{17}^9, A_{19}^{10}, A_{20}^7, and A_{21}^{11} are found to have especially good covering thickness, with the last four currently the thinnest coverings available in their respective dimensions (see Baranovskii 1994, Anzin 2002, and Sikirić, Schürmann, & Vallentin 2008). Note also that $A_7^2 \cong E_7$, $A_7^4 \cong E_7^*$, and $A_8^3 \cong E_8$, each of which is discussed further below.

Note finally that the contact graphs of some of the A_n^r lattices, such as A_9^5 and A_{11}^4, are disjoint nets given by the contact graphs of the A_n and shifted A_n sets of lattice points upon which these glued zero-sum lattices are built [see (2.8)]. Thus, as in the case of D_n^* for $n > 4$ as discussed in §2.3, a *generalized net* may be formed by connecting each node of the unshifted A_n set to the nearest nodes on the shifted A_n set. Again, the resulting net is uninodal, but is not a contact graph of the corresponding sphere packing.

2.5 The E_8 (Gosset), E_7, & E_6 lattices and their duals

The *Gosset lattice* $E_8 \cong E_8^*$, which has a (remarkable) kissing number of $\tau = 240$, may be defined simply as

$$E_8 = D_8^+, \tag{2.9a}$$

and may be constructed via integer linear combination of the columns of the 8×8 basis matrix

$$B_{E_8} = \begin{pmatrix} 2 & -1 & & & & & 0 & 1/2 \\ & 1 & -1 & & & & & 1/2 \\ & & 1 & -1 & & & & 1/2 \\ & & & 1 & -1 & & & 1/2 \\ & & & & 1 & -1 & & -1/2 \\ & & & & & 1 & -1 & -1/2 \\ & & & & & & 1 & -1/2 \\ 0 & & & & & & & -1/2 \end{pmatrix}. \tag{2.9b}$$

The lattice E_7 is defined by restricting E_8, as constructed above, to a 7-dimensional subspace,

$$E_7 = \{(x_1, \ldots, x_8) \in E_8 : x_1 + \ldots + x_8 = 0\}, \tag{2.10a}$$

and may be constructed directly via integer linear combination of the columns of the 8×7 basis matrix

$$B_{E_7} = \begin{pmatrix} -1 & & & & & 0 & 1/2 \\ 1 & -1 & & & & & 1/2 \\ & 1 & -1 & & & & 1/2 \\ & & 1 & -1 & & & 1/2 \\ & & & 1 & -1 & & -1/2 \\ & & & & 1 & -1 & -1/2 \\ & & & & & 1 & -1/2 \\ 0 & & & & & & -1/2 \end{pmatrix}, \qquad \text{with} \qquad \mathbf{n}_{E_7} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}. \tag{2.10b}$$

The dual of the E_7 lattice may be written as

$$E_7^* = E_7 \cup ([1] + E_7), \qquad \text{where} \quad [1] = \begin{pmatrix} 1/4 \\ \vdots \\ 1/4 \\ -3/4 \\ -3/4 \end{pmatrix}, \tag{2.11a}$$

and may be constructed directly via integer linear combination of the columns of the 8×7 basis matrix

$$B_{E_7^*} = \begin{pmatrix} -1 & & & & & 0 & -3/4 \\ 1 & -1 & & & & & -3/4 \\ & 1 & -1 & & & & 1/4 \\ & & 1 & -1 & & & 1/4 \\ & & & 1 & -1 & & 1/4 \\ & & & & 1 & -1 & 1/4 \\ & & & & & 1 & 1/4 \\ 0 & & & & & & 1/4 \end{pmatrix}, \qquad \text{with} \qquad \mathbf{n}_{E_7^*} = \mathbf{n}_{E_7}. \tag{2.11b}$$

The lattice E_6 is defined by further restricting E_7, as defined in (2.10), to a 6-dimensional subspace,

$$E_6 = \{(x_1, \ldots, x_8) \in E_7 : x_1 + x_8 = 0\}, \tag{2.12a}$$

and may be constructed directly via integer linear combination of the columns of the 8×6 basis matrix

$$B_{E_6} = \begin{pmatrix} & & & & 0 & 1/2 \\ -1 & & & & & 1/2 \\ 1 & -1 & & & & 1/2 \\ & 1 & -1 & & & 1/2 \\ & & 1 & -1 & & -1/2 \\ & & & 1 & -1 & -1/2 \\ & & & & 1 & -1/2 \\ 0 & & & & & -1/2 \end{pmatrix}, \qquad \text{with} \qquad N_E = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \\ 0 & 1/2 \\ 0 & 1/2 \\ 0 & 1/2 \\ 0 & 1/2 \\ 0 & 1/2 \\ 1 & 1/2 \end{pmatrix} = \begin{pmatrix} | & | \\ \mathbf{n}_{E_6} & \mathbf{n}_{E_7} \\ | & | \end{pmatrix}. \tag{2.12b}$$

The dual of the E_6 lattice may be written as

$$E_6^* = E_6 \cup ([1] + E_6) \cup ([2] + E_6), \qquad \text{where} \quad [1] = \begin{pmatrix} 0 \\ -2/3 \\ -2/3 \\ 1/4 \\ \vdots \\ 1/4 \\ 0 \end{pmatrix}, \quad [2] = -[1], \tag{2.13a}$$

and may be constructed directly via integer linear combination of the columns of the 8×6 basis matrix

$$B_{E_6^*} = \begin{pmatrix} & & & 0 & 0 & 1/2 \\ -1 & & & & 2/3 & 1/2 \\ 1 & -1 & & & 2/3 & 1/2 \\ & 1 & -1 & & -1/3 & 1/2 \\ & & 1 & -1 & -1/3 & -1/2 \\ & & & 1 & -1/3 & -1/2 \\ & & & & -1/3 & -1/2 \\ 0 & & & & 0 & -1/2 \end{pmatrix}, \qquad \text{with} \qquad N_{E^*} = N_E. \tag{2.13b}$$

2.6 The laminated lattices Λ_n and the closely-related K_n lattices

The lattices in the Λ_n and K_n families can be built up one dimension, or "laminate", at a time, starting from the integer lattice ($\mathbb{Z} \cong \Lambda_1 \cong K_1$), to triangular ($A_2 \cong \Lambda_2 \cong K_2$), to FCC ($A_3 \cong D_3 \cong \Lambda_3 \cong K_3$), all the way up (one layer at a time) to the remarkable Leech lattice ($\Lambda_{24} \cong K_{24}$). Both families of lattices may in fact be extended (but not uniquely) to at least $n = 48$.

The Leech lattice, Λ_{24}, is the unique lattice in $n = 24$ dimensions with a (remarkable) kissing number of $\tau = 196,560$. It may be constructed via integer linear combination of the columns of the 24×24 basis matrix $B_{\Lambda_{24}}$, which is depicted below in the celebrated Miracle Octad Generator (MOG) coordinates (see Curtis 1976 and Conway & Sloane 1998). Further, as in the $E_8 \to E_7 \to E_6$ progression described in §2.5, the Λ_n lattices for $n = 23, 22, \ldots, 1$ may all be constructed by restricting the Λ_{24} lattice to smaller and smaller subspaces via the normal vectors assembled in the matrix N_Λ depicted below[4].

[4]There are, of course, *many* equivalent constructions of Λ_1 through Λ_{23} via restriction of Λ_{24}, and the available literature on the

subject considers these symmetries at length. The convenient form of N_Λ depicted here was deduced, with some effort, from Figure 6.2 of Conway & Sloane (1998).

$$B_{\Lambda_{24}} = \frac{1}{\sqrt{8}} \begin{pmatrix}
8 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 4 & 4 & 4 & 2 & 4 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & & & -3 \\
 & 4 & & & & & & 2 & & & & 2 & & 2 & & & & & & 2 & 2 & & 1 \\
 & & 4 & & & & & 2 & & & & 2 & & & 2 & & & 2 & & & 2 & & 1 \\
 & & & 4 & & & & 2 & & & & 2 & & & & 2 & & & 2 & & 2 & & 1 \\
 & & & & 4 & & & 2 & & & & & & 2 & 2 & 2 & & 2 & 2 & 2 & 2 & & 1 \\
 & & & & & 4 & & 2 & & & & & & 2 & & & & & 2 & & & & 1 \\
 & & & & & & 4 & 2 & & & & & & & 2 & & & & & 2 & & & 1 \\
 & & & & & & & 2 & & & & & & & & 2 & & 2 & & & & & 1 \\
 & & & & & & & & 4 & & & 2 & & 2 & 2 & 2 & & 2 & 2 & 2 & 2 & 2 & 1 \\
 & & & & & & & & & 4 & & 2 & & 2 & & & & 2 & & & & 2 & 1 \\
 & & & & & & & & & & 4 & 2 & & & & & & & 2 & & & & 1 \\
 & & & & & & & & & & & 2 & & & & 2 & & & & 2 & & & 1 \\
 & & & & & & & & & & & & 4 & 2 & 2 & 2 & & & & & 2 & 2 & 1 \\
 & & & & & & & & & & & & & 2 & & & & & & & 2 & & 1 \\
 & & & & & & & & & & & & & & 2 & & & & & & & 2 & 1 \\
 & & & & & & & & & & & & & & & 2 & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & 4 & 2 & 2 & 2 & 2 & 2 & 1 \\
 & & & & & & & & & & & & & & & & & 2 & & & 2 & & 1 \\
 & & & & & & & & & & & & & & & & & & 2 & & & 2 & 1 \\
 & & & & & & & & & & & & & & & & & & & 2 & & & 1 \\
 & & & & & & & & & & & & & & & & & & & & 2 & 2 & 1 \\
 & & & & & & & & & & & & & & & & & & & & 2 & & 1 \\
 & & & & & & & & & & & & & & & & & & & & & 2 & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & 1
\end{pmatrix}$$

$$\begin{pmatrix}
 & & 1 & & & & & & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & & & & & & & & \\
\end{pmatrix}$$

Thus, the Λ_{23} lattice is obtained from the points of the Λ_{24} lattice in $\mathbb{R}^{24}$ (which themselves are generated via integer linear combination of the columns of $B_{\Lambda_{24}}$) which lie in the 23-dimensional subspace orthogonal to $\mathbf{n}_{\Lambda_{23}}$. Similarly, the Λ_{22} lattice is obtained from the points of the Λ_{24} lattice which lie in the 22-dimensional subspace orthogonal to both $\mathbf{n}_{\Lambda_{23}}$ and $\mathbf{n}_{\Lambda_{22}}$, etc. Noting the block diagonal structure of N_Λ, it follows that Λ_n may be constructed using the basis matrix, denoted B_{Λ_n}, given by the $n \times n$ submatrix in the upper-left corner of $B_{\Lambda_{24}}$ for any $n \in \mathsf{N}_1 = \{21,20,16,9,8,5,4\}$. For the remaining dimensions, $n \in \mathsf{N}_2 = \{19,18,17,15,14,13,12,11,10,7,6,3,2,1\}$, Λ_n may be constructed via the appropriate restriction of the lattice generated by the next larger basis matrix in the set N_1; for example, Λ_{14} may be constructed in $\mathbb{R}^{16}$ via restriction of the lattice generated by the basis matrix $B_{\Lambda_{16}}$ to the subspace normal to the vectors (in $\mathbb{R}^{16}$) given by the first 16 elements of $\mathbf{n}_{\Lambda_{15}}$ and $\mathbf{n}_{\Lambda_{14}}$.

A similar sequence of lattices, denoted K_n, may be constructed via restriction of the Leech lattice (generated via $B_{\Lambda_{24}}$) in a similar fashion (for details, see Figure 6.3 of Conway & Sloane 1998). Lattices from the Λ_n and/or K_n families have the maximal packing densities and kissing numbers amongst all lattices for the entire range considered here, $1 \leq n \leq 24$. Note that the Λ_n and K_n families are not equivalent in the range $7 \leq n \leq 17$, with Λ_n being superior to K_n by all four metrics introduced in the Preface at most values of n in this range, except for the narrow range $11 \leq n \leq 13$, where in fact K_n has a slight advantage. Note also that there is some flexibility in the definition of the lattices Λ_{11}, Λ_{12}, and Λ_{13}; the branch of the Λ_n family considered here is that which maximizes the kissing number τ in this range of n, and thus the corresponding lattices are denoted $\Lambda_{11}^{\max}$, $\Lambda_{12}^{\max}$, and $\Lambda_{13}^{\max}$. Note that K_{12} is referred to as the Coxeter-Todd lattice and Λ_{16} is referred to as the Barnes-Wall lattice.

2.7 Numerically-generated lattices with thin coverings for $n = 6$ to 15

Recall from §2.1 that an n-dimensional real lattice may be defined as an integer linear combination of a set of n basis vectors $\mathbf{b}^i$ defined in $\mathbb{R}^{n+m}$ for $m \geq 0$; that is, any lattice point may be written as

$$\mathbf{x} = y_1\mathbf{b}^1 + y_2\mathbf{b}^2 + \ldots + y_n\mathbf{b}^n = B\mathbf{y},$$

where the elements $\{y_1, \ldots, y_n\}$ of the vector $\mathbf{y}$ are taken as integers. The square of the distance of any lattice point from the origin is thus given by $f(\mathbf{y}) = \mathbf{y}^T A\,\mathbf{y}$, where $A \triangleq B^T B$ is known as the *Gram matrix* associated with the lattice in question, and the function $f(\mathbf{y})$ is referred to as the corresponding *quadratic form* [note that each term of $f(\mathbf{y})$ is quadratic in the elements of $\mathbf{y}$]. All of the lattices studied thus far, when scaled appropriately, are characterized by Gram matrices with *integer elements*, and thus their corresponding quadratic forms $f(\mathbf{y})$ have integer coefficients (and are thus referred to as *integral quadratic forms*).

There is particular mathematical interest in discovering (or generating numerically) both lattice and non-lattice packings which minimize covering thickness and/or packing density. The numerical approach to this problem studied in Schürmann & Vallentin (2006) and Sikirić, Schürmann, & Vallentin (2008) has generated new lattices in dimensions $n = 6$ to 15 with the thinnest covering thicknesses known amongst all lattices[5]. The lattice so generated in dimension 7 happens to correspond to an integral quadratic form, but the others, apparently, do not.

[5] Gram matrices A corresponding to these 10 lattices (denoted L_6^{c1}, L_7^c, L_8^c, …, L_{15}^c) are available at
http://fma2.math.uni-magdeburg.de/~latgeo/covering_table.html
(nonunique) basis matrices B corresponding to each of these lattices may be generated simply by taking the Cholesky decomposition of the corresponding Gram matrix, as $A = B^T B$.