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A NUMERICALLY TRACTABLE GLOBAL FRAMEWORK FOR THE FEEDBACK CONTROL OF BOUNDARY-LAYER PERTURBATIONS

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ABSTRACT

We present a global framework for the feedback control of a large class of spatially-developing boundary-layer flow systems. The systems considered are (approximately) parabolic in the spatial coordinate x . This facilitates the application of a range of established feedback control theories which are based on the solution of differential Riccati equations which march over a finite horizon in x (rather than marching in t , as customary). However, unlike systems which are parabolic in time, there is no causality constraint for the feedback control of systems which are parabolic in space; that is, downstream information may be used to update the controls upstream. Thus, a particular actuator may be used to neutralize the effects of a disturbance which actually enters the system downstream of the actuator location. In the present study, a numerically-tractable feedback control strategy is proposed which takes advantage of this special capability of feedback control rules in the spatially-parabolic setting in order to minimize a globally-defined cost function in an effort to maintain laminar boundary-layer flow.

INTRODUCTION

This paper considers the feedback estimation and control of small, spatially-developing, three-dimensional perturbations to a thin laminar boundary layer in a viscous wall-bounded flow. Control is applied via a blowing/suction distribution over a portion of the wall, and state estimation is accomplished via measurements of skin friction and pressure distributed over the same region. The wall-normal direction is taken to be y and the leading edge of the surface, which might be blunt, is near the line defined by $x = y = 0$; the wall thus lies in the half plane $\{y = 0, x \gtrsim 0\}$. In the special case of an unswept flat plate, the streamwise direction is x and the spanwise direction is z . More generally, the leading edge of the surface over which the boundary layer develops may be swept, and the surface may be inclined and/or curved in the

x - y plane. The curvilinear coordinate system is fitted to the body such that the surface is defined by $\{y = 0, x \gtrsim 0\}$ even when the leading edge is swept and the surface is curved. Special cases of interest included in the framework presented here include the stabilization of the Blasius, Falkner-Skan, Falkner-Skan-Cooke, and Görtler families of boundary-layer flows.

An important characteristic of laminar systems of this type, which fall under the classic "boundary-layer assumption"¹, is that they are essentially independent of time², and the equations that govern them, subject to the correct approximations, are *parabolic in x* . Further generalizations to the framework presented here, such as accounting for heat transfer to or from the surface, are straightforward extensions as long as the boundary-layer assumption remains valid.

There is a large body of work in the fluids literature on iterative (adjoint-based) control optimization strategies for boundary-layer flow systems. These strategies are appropriate for open-loop control optimizations, and are beginning to see successful applications in this regard. For recent reviews of this line of research, see, e.g., Walter et al. (2001), Cathalifaud & Luchini (2000), and Pralits et al. (2000). However, it is computationally quite difficult (if not impossible) to apply iterative, adjoint-based control optimization strategies in the closed-loop setting to neutralize the effects of the random flow disturbances that arise in nature. For such problems, feedback control strategies which can respond quickly and in a coordinated fashion to measurements of the flow system are necessary.

¹The boundary layer assumption is that the boundary-layer thickness is much smaller than the streamwise length scales in the system, and that the time scale of the external perturbations to the system are large with respect to the boundary-layer thickness divided by the freestream velocity (see, e.g., Schlichting 1978).

²Time variations in the system model are easily accounted for by gradual variation of the inflow conditions and the external disturbances.

There is a large body work in the controls literature on the feedback estimation and control of systems which are parabolic in time. Of particular interest for non-normal systems, such as those often encountered in fluid mechanics, is the fact that $\mathcal{H}_2/\mathcal{H}_\infty$ control theory, which is quite well suited to such systems, is now well understood for both infinite-horizon and finite-horizon control problems, and is discussed in detail in standard textbooks (see, e.g., Green & Limebeer 1995). Applications of this and related feedback control theories are now finding their way into many control problems in fluid mechanics. Though subtle issues related to the infinite dimension and inflow/outflow conditions make the application of established feedback control strategies to such systems nontrivial, significant progress has been made in recent years. For a recent review of this line of research, see Bewley (2001). The present paper develops a closed-loop, Riccati-based *feedback* control strategy (as opposed to an open-loop, adjoint-based control *optimization* strategy) to a spatially-developing boundary layer flow system.

Control strategies for systems which evolve parabolically in time must be *causal*; that is, they must depend only on present and past measurements of the flow. However, control strategies for systems which evolve parabolically in space are not limited by such a constraint; the control at a particular actuator location may depend on measurements taken both upstream and downstream. Thus, to exploit the additional measurement information available in this setting, a different set of tools is needed for this problem beyond the standard LQG ($\mathcal{H}_2$) framework and “robustifying” extensions thereof ($\mathcal{H}_\infty$, LTR, etc.). In fact, the necessary control theory for the present problem was essentially laid out by Anderson & Moore (1979) and Middleton & Goodwin (1990). The present paper discusses the additional considerations necessary to apply these tools to boundary-layer flows.

Unlike recent efforts to develop decentralized feedback control strategies for boundary-layer flows, which depend only upon flow measurements and state estimates in the immediate vicinity of any given actuator, the present approach sacrifices localization of the feedback rules in the streamwise coordinate in order to achieve possibly significant performance improvements over that possible with localized strategies. Performance comparisons will be conducted in future numerical studies; the purpose of the present paper is simply to present a numerically-tractable framework for a global strategy for the feedback control of boundary-layer flow systems.

GOVERNING EQUATIONS

Based on the dimensional coordinates $\{x^*, y^*, z^*\}$, velocities $\{u^*, v^*, w^*\}$, and pressure p^* , we define the dimensionless quantities $x = x^*/L$, $\{y, z\} = \{y^*, z^*\}/\delta$, $u = u^*/U_o$, $\{v, w\} = \{v^*, w^*\}Re_\delta/U_o$, and $p = p^*Re_\delta^2/(\rho U_o^2)$, where U_o is the freestream velocity, ρ is the density, μ is the viscosity, $\nu = \mu/\rho$ is the kinematic viscosity, L is a reference streamwise

length, $\delta = \sqrt{Lv/U_o}$ is a reference boundary layer thickness, and $Re_\delta = U_o \delta/\nu$ is a reference Reynolds number. Also, from the dimensional radius of curvature r^* of the surface in the x - y plane, we define the dimensionless curvature parameter $\varepsilon = \delta/|r^*|$, the Görtler number $G = Re_\delta \sqrt{\varepsilon}$, and a sign function s such that $s = 0$ corresponds to a flat wall, $s = 1$ corresponds to a concave wall, and $s = -1$ corresponds to a convex wall.

In order to apply the boundary-layer approximation and to develop a linear set of equations governing small perturbations to the nominal (undisturbed) boundary-layer flow, we make the following assumptions:

A1: $\delta \ll L$ (i.e., $Re_\delta \gg 1$);

A2: $\delta \ll |r^*|$ (i.e., $\varepsilon \ll 1$);

A3: $G \lesssim O(1)$;

A4: the nominal (undisturbed) flow is laminar and steady.

Note that the boundary-layer approximation of the Navier-Stokes equations is not valid in the vicinity of the leading edge. The present work avoids this singularity by considering the evolution of the system only over the interval over which the control is applied, which we define as $x_0 \leq x \leq L$, where $x_0 > 0$. In order to develop control strategies which are not sensitive to errors in the modeling of the flow upstream of x_0 , we will seek control strategies which are insensitive to small errors in the nominal inflow velocity profile.

Though not necessary for the application of the present control approach, it is convenient to approximate the nominal boundary layer flow $\{U(x, y), V(x, y), W(x, y)\}$ by a profile of the Blasius/Falkner-Skan-Cooke/Görtler family (see, e.g., Cooke 1950). Similarity solutions of this commonly-occurring class of boundary-layer flows may be found from the coupled ODEs

$$\begin{aligned} f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) &= 0, & g'' + \frac{m+1}{2} f g' &= 0, \\ f(0) = f'(0) = 0, \quad f'(\infty) &\rightarrow 1, & g(0) = 0, \quad g(\infty) &\rightarrow 1, \end{aligned}$$

by defining $U_0 = x^m$, $\eta = y\sqrt{U_0/x}$, $U = U_0 f'(\eta)$, $W = W_0 g(\eta)$, and $V = \sqrt{U_0/x}[(1-m)\eta f' - (1+m)f]/2$. Alternatively, for systems in which, e.g., the curvature of the wall changes gradually as a function of x (as with the flow over a typical airfoil), the nominal boundary-layer flow profile $\{U(x, y), V(x, y), W(x, y)\}$ may be found via straightforward numerical integration of the steady-state boundary-layer equations over the appropriate geometry.

Small three-dimensional perturbations to the nominal flow, $\{u(x, y, z), v(x, y, z), w(x, y, z)\}$, are governed by the linearized Navier-Stokes equation. As the system governing these perturbations is linear and homogeneous in z , we may decouple the various spanwise modes of this system by taking the Fourier transform of all perturbation variables with spanwise variation (namely, the state, the controls, the measurements, and the disturbances) in the z direction (see, e.g., Bewley & Liu 1998). In

the present discussion, we therefore consider a particular Fourier mode of the flow perturbations, and assign a variation in z of $\exp(-i\beta z)$ to all of these variables. Once the control problem is solved for a series of spanwise wavenumbers, inverse Fourier transform of the feedback gains should lead to feedback convolution kernels which are spatially localized in the spanwise coordinate (Bewley 2001, Bamieh et al. 2002). Such localization in the spanwise coordinate of the feedback convolution kernels which result from the solution of this control problem will be explored in future work.

Following the analysis of Hall (1985), under the boundary-layer assumptions itemized above, the linearized, nondimensional equations for the flow perturbations reduce to

$$\begin{aligned} (Uu)_x + Vu_y + U_y v - i\beta W u - u_{yy} + \beta^2 u &= 0, \\ Uv_x + Vxu + (Vv)_y + p_y - i\beta W v + 2sG^2 Uu - v_{yy} + \beta^2 v &= 0, \\ UW_x + Wxu + Vw_y + W_y v - i\beta p - i\beta W w - w_{yy} + \beta^2 w &= 0, \\ u_x + v_y - i\beta w &= 0, \end{aligned} \quad (1)$$

with the boundary and initial conditions :

$$\begin{aligned} u = w = 0, \quad v = v_w(x), \quad \text{at } y = 0, \\ u = v = w = 0, \quad \text{at } y = \infty, \\ \{u, v, w\} = \{u_0, v_0, w_0\}, \quad \text{at } x = x_0, \end{aligned} \quad (2)$$

where $v_w(x)$ is the control velocity of blowing and suction distributed over the wall on the strip $x_0 < x < L$. The purpose of the control in this problem is to keep the flow perturbations sufficiently small that transition to turbulence is inhibited.

Define the normal vorticity $\eta^* = \partial u^* / \partial z^* - \partial w^* / \partial x^*$ and the corresponding dimensionless form $\eta = -i\beta u - w_x / Re_\delta^2$. We now combine the governing equations (1) in such a way as to determine a set of two coupled equations for the perturbation components of the normal velocity and normal vorticity. The first of these equations is found by taking the Laplacian of the second component of the momentum equation, substituting the expression for Δp found by taking the divergence of the momentum equation, and applying continuity. The second of these equations is found by taking the normal component of the curl of the momentum equation. Defining $D^k = \partial^k / \partial y^k$ the result is

$$\begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & \tilde{E}_{22} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix}, \quad (3)$$

where

$$\begin{aligned} \tilde{E}_{11} &= U(D^2 - \beta^2) - U_{yy} \\ \tilde{E}_{12} &= -(2i/\beta)[U_{xy} + U_x D^1] \\ \tilde{E}_{22} &= -U \end{aligned}$$

$$\begin{aligned} \tilde{A}_{11} &= -[(V_{yy} - \beta^2 V)D^1 + V_{yyy} + V_y(D^2 - \beta^2) \\ &\quad + VD^3 - D^4 + 2\beta^2 D^2 - \beta^4 + i\beta W_{yy} - i\beta W D^2 + i\beta^3 W] \\ \tilde{A}_{12} &= -\frac{i}{\beta}[V_{xyy} - V_x(D^2 + \beta^2) + 2i\beta(W_x D^1 - W_{xy}) - 2\beta^2 G^2 U] \\ \tilde{A}_{21} &= -i\beta U_y \\ \tilde{A}_{22} &= [U_x + VD^1 - D^2 + \beta^2 - i\beta W]. \end{aligned}$$

STATE-SPACE FORMULATION: DISCRETIZATION IN y

We now perform a discretization of the system in the y coordinate on a finite number of discretization points with the appropriate grid stretching. Let $\{\mathbf{v}, \boldsymbol{\eta}\}$ denote the spatial discretizations of $\{v, \eta\}$ on the interior of the domain. The derivative operators D^k may be approximated in this discretization using any of a variety of techniques, such as finite differences, Padé, Chebyshev, etc. Define the matrices $\{\hat{E}_{11}, \hat{E}_{12}, \hat{E}_{22}, \hat{A}_{11}, \hat{A}_{12}, \hat{A}_{21}, \hat{A}_{22}\}$ as the spatial discretizations of $\{\tilde{E}_{11}, \tilde{E}_{12}, \tilde{E}_{22}, \tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{21}, \tilde{A}_{22}\}$ on the interior of the domain using the chosen technique, and the vectors $\{\mathbf{e}_{11}, \mathbf{a}_{11}\}$ to denote the influence of the normal velocity at the wall on the left-hand side and right-hand side of the v component of the discretization of (3). Using these discrete forms, it is straightforward to express (3) in the state-space form

$$\begin{aligned} \mathbf{q}_x &= A\mathbf{q} + B\phi, \\ \mathbf{q} &= \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\eta} \end{pmatrix}, \quad A = \begin{pmatrix} \hat{E}^{-1}\hat{A} & \hat{E}^{-1}\mathbf{a} \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -\hat{E}^{-1}\mathbf{e} \\ 1 \end{pmatrix} \\ \hat{E} &= \begin{pmatrix} \hat{E}_{11} & \hat{E}_{12} \\ 0 & \hat{E}_{22} \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} \mathbf{e}_{11} \\ 0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \mathbf{a}_{11} \\ 0 \end{pmatrix}. \end{aligned} \quad (4)$$

The control variable in this formulation is $\phi = dv_w/dx$.

SYSTEM DISTURBANCES AND DISCRETIZATION IN x

To account for external system disturbances and modeling uncertainties, we now modify the state equation (4) by adding disturbances $\mathbf{w}$ to the right-hand side:

$$\mathbf{q}_x = A\mathbf{q} + B\phi + D\mathbf{w}, \quad (5)$$

where the disturbance vector $\mathbf{w}$ depends on the spatial coordinate x . We desire to develop a global strategy in which the control $\phi(x)$ may actually respond to disturbances $\mathbf{w}(x)$ acting over the entire domain under consideration $x_0 \leq x \leq L$. To facilitate this in the standard (causal) setting, we first discretize the system in x , then define an augmented state

$$\mathbf{q}_k^a = \begin{pmatrix} \mathbf{q}_k \\ \mathbf{q}_k^w \end{pmatrix} \quad (6)$$

at each station $x_k = x_0 + k\Delta$, $k = 0, \dots, N$, where $\Delta = (L - x_0)/N$ represents the grid spacing in x , $\mathbf{q}_k = \mathbf{q}(x_k)$, $\mathbf{w}_k = \mathbf{w}(x_k)$, and

$$\mathbf{q}_0^w = \begin{pmatrix} \mathbf{w}_0 \\ \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_N \end{pmatrix}, \quad \mathbf{q}_1^w = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_N \\ 0 \end{pmatrix}, \quad \dots \quad \mathbf{q}_N^w = \begin{pmatrix} \mathbf{w}_N \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that the augmented state $\mathbf{q}_k^a$ at a particular streamwise station x_k need only include the disturbances entering the system downstream of that location, as the influence of the disturbances upstream are accounted for in $\mathbf{q}_k$. Note also that we can express the evolution of $\mathbf{q}_k^w$ in the discrete state-space form

$$\mathbf{q}_{k+1}^w = A^d \mathbf{q}_k^w, \quad A^d = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}, \quad (7)$$

where the relation between $\mathbf{w}_k$ and $\mathbf{q}_k^w$ is

$$\mathbf{w}_k = M^w \mathbf{q}_k^w, \quad M^w = (I \ 0 \ \dots \ 0). \quad (8)$$

By combining equations (5), (7), and (8), we can obtain a state-space formulation for the augmented state $\mathbf{q}^a$. However, the inherently discrete nature of the evolution of our disturbance model $\mathbf{q}^w$ compels us to first derive a discrete formulation of the state equation (5). To accomplish this, we approximate $\{A, B, \mathbf{q}, \phi\}$ with $\{A_k, B_k, \mathbf{q}_k, \phi_k\}$ over the interval $x_k \leq x < x_{k+1}$ for each value of k , where, e.g., $A_k = A(x_k)$. Using this approximation (commonly referred to as a “zero-order hold”), we may express (5) in the following “delta form” (Middleton and Goodwin, 1990):

$$\delta \mathbf{q}_k = \Omega_k A_k \mathbf{q}_k + \Omega_k B_k \phi_k + \Omega_k D_k \mathbf{w}_k, \quad (9)$$

where $\Omega_k = (1/\Delta) \int_0^\Delta \exp(A_k \tau) d\tau$ and $\delta \mathbf{q}_k = (\mathbf{q}_{k+1} - \mathbf{q}_k)/\Delta$. Note in particular that $\Omega_k \rightarrow I$ as $\Delta \rightarrow 0$, and thus the discrete-in- x relation (9) tends towards the continuous-in- x relation (5) as the grid is refined. This behaviour of the δ -formulation also follows for the Riccati and Lyapunov equations that arise in the control and estimation problems in the following sections, and is an appealing characteristic of this particular discrete formulation. Note that the calculation of the matrix exponential necessary to determine Ω_k can be performed with any of at least 19 dubious techniques (Moler and Van Loan, 1978). One of the least dubious of these techniques is the so-called scaling and squaring method.

Combining (9), (7), and (8), we finally obtain a discrete, causal state-space formulation for the augmented state, to which standard control theories may be applied:

$$\delta \mathbf{q}_k^a = A_k^a \mathbf{q}_k^a + B_k^a \phi_k \quad (10)$$

where $A_k^a = \begin{pmatrix} \Omega_k A_k & \Omega_k D_k M^w \\ 0 & A^d \end{pmatrix}$ and $B_k^a = \begin{pmatrix} \Omega_k B_k \\ 0 \end{pmatrix}$.

OPTIMAL CONTROL FOR NONCAUSAL SYSTEMS

In the original PDE setting, our control objective may be written as finding a feedback control rule which minimizes the cost function

$$J = \int_{x_0}^L \left[\int_0^\infty u^* u dy + \alpha_1^2 v_w^* v_w + \alpha_2^2 \phi^* \phi \right] dx.$$

Discretizing in x and y , the cost function may be approximated by

$$J = \sum_{i=0}^N \Delta \left[(\mathbf{q}^a)_i^* Q^a \mathbf{q}_i^a + \alpha_2^2 \phi_i^* \phi_i \right], \quad (11)$$

where $Q^a = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} I_s/\beta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_1^2 \end{pmatrix}$, and I_s is a diagonal matrix with the corresponding local grid spacing on the elements of the diagonal.

Note that the technique of augmenting the initial state with the disturbances entering the entire system [see (6)] facilitated the conversion of the noncausal problem described in the introduction into the causal problem represented by (10). Together with the control objective (11), a feedback control rule of the form

$$\phi_k = -\mathbf{K}_{k+1} \mathbf{q}_k^a \quad (12)$$

may be found directly using standard “discrete-time” optimal control theory. In fact, as discussed in Bitmead et al. (1990), the Riccati equation associated with this control problem may be partitioned in a convenient fashion by defining

$$\begin{aligned} \mathbf{K}_k &= (\alpha_2^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k)^{-1} B_k^* \Omega_k^* (K_k^1 \ K_k^2), \\ K_k^1 &= \Sigma_k^{11} (I + \Delta \Omega_k A_k), \\ K_k^2 &= \Delta \Sigma_k^{11} \Omega_k D_k M^w + \Sigma_k^{12} (I + \Delta A_k^d), \end{aligned} \quad (13)$$

where Σ_k^{11} and Σ_k^{12} solve the Riccati and Lyapunov equations

$$\begin{aligned}\bar{\delta}\Sigma_k^{11} &= Q + A_k^* \Omega_k^* \Sigma_k^{11} + \Sigma_k^{11} \Omega_k A_k + \Delta A_k^* \Omega_k^* \Sigma_k^{11} \Omega_k A_k - \\ &\quad (K_k^1)^* \Omega_k B_k [\alpha_2^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k]^{-1} B_k^* \Omega_k^* K_k^1, \\ \bar{\delta}\Sigma_k^{12} &= A_k^* \Omega_k^* \Sigma_k^{12} + [I + \Delta A_k^* \Omega_k^*] [\Sigma_k^{11} \Omega_k D_k M^w + \Sigma_k^{12} A^d] - \\ &\quad (K_k^1)^* \Omega_k B_k [\alpha_2^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k]^{-1} B_k^* \Omega_k^* K_k^2,\end{aligned}\quad (14)$$

where $\bar{\delta}\Sigma_k = -(\Sigma_k - \Sigma_{k-1})/\Delta$. As $\Delta \rightarrow 0$, equations (14) tend towards the corresponding continuous Riccati and Lyapunov equations (cf. Middleton & Goodwin, 1990).

Finally, by combining (12) and (10), we can express ϕ_k as a simple function of the initial augmented state vector $\mathbf{q}_0^a$:

$$\phi_k = \mathbf{K}_{k+1}^0 \mathbf{q}_0^a, \quad (15)$$

where $\mathbf{K}_{k+1}^0 = -\mathbf{K}_{k+1} \prod_{i=0}^{k-1} (A_i^a - \mathbf{K}_{i+1} B_i^a)$.

OPTIMAL ESTIMATION/SMOOTHING

By (15), we see that we can express the optimal control distribution on $x_0 < x < L$ which minimizes the globally-defined cost function $\mathcal{J}$ as a simple function of the upstream flow perturbation $\mathbf{q}_0$ and the system disturbances $\mathbf{w}(x)$ between x_0 and L . The task which remains is to find a simple way to obtain a good estimate of $\mathbf{q}_0^a$ based on the available measurements at the wall.

Defining the vector $\boldsymbol{\mu}$ as the measurement noise, the measurements of the streamwise and spanwise skin friction and pressure distributions over the wall may be written as

$$\mathbf{y}(x) = \begin{pmatrix} \frac{\partial u}{\partial y} \Big|_{\text{wall}}(x) \\ \frac{\partial w}{\partial y} \Big|_{\text{wall}}(x) \\ p|_{\text{wall}}(x) \end{pmatrix} + \boldsymbol{\mu}, \quad (16)$$

Note that (from Bewley & Protas 2002, applying the nondimensionalization discussed previously) we may write

$$\begin{aligned}\frac{\partial \eta}{\partial y} \Big|_{\text{wall}} &= -i\beta \frac{\partial u}{\partial y} \Big|_{\text{wall}} - \frac{1}{Re_\delta^2} \frac{\partial}{\partial x} \frac{\partial w}{\partial y} \Big|_{\text{wall}}, \\ \frac{\partial^2 v}{\partial y^2} \Big|_{\text{wall}} &= -\frac{\partial}{\partial x} \frac{\partial u}{\partial y} \Big|_{\text{wall}} + i\beta \frac{\partial w}{\partial y} \Big|_{\text{wall}}, \\ \frac{\partial^3 v}{\partial y^3} \Big|_{\text{wall}} &= \left(\beta^2 - \frac{1}{Re_\delta^2} \frac{\partial^2}{\partial x^2} \right) p|_{\text{wall}} - \frac{\partial U}{\partial y} \Big|_{\text{wall}} \frac{\partial v_w}{\partial x}.\end{aligned}\quad (17)$$

By neglecting the terms in $1/Re_\delta^2$ in (17) we can express the skin friction and pressure at the wall as:

$$\begin{pmatrix} \frac{\partial u}{\partial y} \Big|_{\text{wall}} \\ \frac{\partial w}{\partial y} \Big|_{\text{wall}} \\ p|_{\text{wall}} \end{pmatrix} = Z \begin{pmatrix} \frac{\partial \eta}{\partial y} \Big|_{\text{wall}} \\ \frac{\partial^2 v}{\partial y^2} \Big|_{\text{wall}} \\ \frac{\partial^3 v}{\partial y^3} \Big|_{\text{wall}} \end{pmatrix} + N\phi, \quad \text{where}$$

$$Z = \begin{pmatrix} i/\beta & 0 & 0 \\ 1/\beta^2 \partial/\partial x & -i/\beta & 0 \\ 0 & 0 & 1/\beta^2 \end{pmatrix}, \quad N = \begin{pmatrix} 0 \\ 0 \\ \partial U/\partial y|_{\text{wall}} \end{pmatrix} \quad (18)$$

Using the relations (16) and (18), we can approximate the vector of the wall measurements $\mathbf{y}$ as a function of the discrete state vector $\mathbf{q}$, the control variable ϕ , and the measurement noise $\boldsymbol{\mu}$:

$$\mathbf{y} = ZM\mathbf{q} + N\phi + \boldsymbol{\mu} \quad \text{where} \quad M = \begin{pmatrix} \frac{\delta^1}{\delta y^1} \Big|_w & 0 \\ 0 & \frac{\delta^2}{\delta y^2} \Big|_w \\ 0 & \frac{\delta^3}{\delta y^3} \Big|_w \end{pmatrix} \quad (19)$$

and the notation $\delta^k/\delta y^k|_w$ denotes the discretization of the k 'th derivative operator evaluated at the wall. Applying the definition of the augmented state $\mathbf{q}^a$, we may write (19) as

$$\mathbf{y}_k = M^a \mathbf{q}_k^a + N_k \phi_k + \boldsymbol{\mu}_k \quad (20)$$

where $M^a = (ZM \ 0)$. We now define the notation $\hat{\mathbf{q}}_{k|m}^a = \hat{\mathbf{q}}^a(x_k|x_m)$ to denote the estimate of $\mathbf{q}^a(x_k)$ based on the measurements $\mathbf{y}(x)$ from $x_0 \leq x \leq x_m$. Our aim is to calculate an estimate of $\mathbf{q}_0^a$ based on the measurements $\mathbf{y}(x)$ for $x_0 \leq x \leq x_N = L$ (i.e. $\hat{\mathbf{q}}_{0|N}^a$). This is a ‘‘smoothing’’ problem, and, given the correct manipulations, can be solved using a Kalman filter. To solve this problem, we first substitute the value of ϕ_k obtained in (12) into the equations (10) and (20). Defining $F_k = A_k^a - B_k^a \mathbf{K}_{k+1}$ and $H_k = M_k^a - N_k \mathbf{K}_{k+1}$, we have

$$\begin{aligned}\delta \mathbf{q}_k^a &= F_k \mathbf{q}_k^a \\ \mathbf{y}_k &= H_k \mathbf{q}_k^a + \boldsymbol{\mu}_k.\end{aligned}\quad (21)$$

Applying Kalman filter theory to the system (21), we obtain the following evolution equation for the estimate $\hat{\mathbf{q}}_{k|k-1}^a$

$$\delta \hat{\mathbf{q}}_{k|k-1}^a = F_k \hat{\mathbf{q}}_{k|k-1}^a + L_k [\mathbf{y}_k - H_k \hat{\mathbf{q}}_{k|k-1}^a], \quad (22)$$

where $L_k = (\Delta F_k + I) P_k^{11} H_k^* [\Delta H_k P_k^{11} H_k^* + C_\mu]^{-1}$ and P_k^{11} is solution of the Riccati equation $\delta P_k^{11} = P_k^{11} F_k^* + F_k P_k^{11} + \Delta F_k P_k^{11} F_k^* - L_k [\Delta H_k P_k^{11} H_k^* + C_\mu] L_k^*$. Our problem actually differs slightly from the filtering problem (22). In particular, the information we want to reconstruct, $\mathbf{q}_0^a$, must be obtained from measurements taken on $x_0 \leq x \leq x_N$. In other words, we seek to determine the value of $\hat{\mathbf{q}}_{0|N}^a$, not the value of $\hat{\mathbf{q}}_{N+1|N}^a$, which can be obtained from (22). As in Anderson and Moore (1979), $\hat{\mathbf{q}}_{0|N}^a$ can be easily derived from the filter problem presented above by initializing this estimate with $\hat{\mathbf{q}}_{0|0}^a = 0$ and marching the discrete equation

$$\hat{\mathbf{q}}_{0|k}^a = \hat{\mathbf{q}}_{0|k-1}^a + \Delta P_k^{12} H_k^* [\Delta H_k P_k^{11} H_k^* + C_\mu]^{-1} [\mathbf{y}_k - H_k \hat{\mathbf{q}}_{k|k-1}^a]$$

from $k = 0$ to $k = N$, where P_k^{12} satisfies the Lyapunov equation

$$\delta P_k^{12} = P_k^{12} (F_k - L_k H_k)^*.$$

We thus obtain $\hat{\mathbf{q}}_{0|N}^a$, which is the best approximation possible of the initial augmented state $\mathbf{q}_0^a$ given all of the measured data on $x_0 \leq x \leq L$. This estimate of the augmented state at x_0 may then be combined with the control relationship (15) to determine the optimal control based on the available noisy measurements.

CONCLUSIONS

The primary challenge in the application of Riccati-based feedback control strategies to fluid-mechanical systems is the enormous state dimension which is necessary to capture such systems with an adequate degree of fidelity. The state dimension necessary to resolve such systems typically renders Riccati-based control strategies numerically unfeasible, and open-loop model reduction strategies are highly prone to misrepresentation of the relevant dynamics of the fluid system, effectively “losing the baby with the bathwater”.

In flow systems with two directions of spatial homogeneity (such as channel flows), the linearized system model may be made approachable with Riccati-based feedback control strategies by decoupling the various streamwise and spanwise modes of the problem using Fourier-based approaches (Bewley & Liu 1998). Linearized boundary-layer systems, however, have only one direction of spatial homogeneity.

The present paper proposes a new, Riccati-based feedback control strategy which leverages the fact that linearized boundary-layer systems develop parabolically in the streamwise coordinate. Taking advantage of this property, numerically-tractable control and estimation algorithms have been proposed which target the reduction of a globally-defined cost function with control feedback while only requiring the solution of Riccati equations related to system models which are spatially-discretized in a single coordinate direction (y). Such Riccati

equations are computationally tractable; numerical results of this formulation will be presented in future work.

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