Fluid Mixing by Feedback in Poiseuille Flow[1]

Ole Morten Aamo[2,3], Miroslav Krstić[2] and Thomas R. Bewley[2]

Abstract

We address the problem of enhancing mixing by means of boundary feedback control in 2D channel flow. This is done by first designing feedback control strategies for the stabilization of the parabolic equilibrium flow, then applying this feedback with the sign of the input reversed. The result is enhanced instability of the parabolic equilibrium flow, which leads rapidly to highly complex flow patterns. Simulations of the deformation of dye blobs positioned in the flow indicate (qualitatively) that effective mixing is obtained for small control effort as compared with the nominal (uncontrolled) flow. A mixedness measure P_ε is constructed to quantify the mixing observed, and is shown to be significantly enhanced by the application of the destabilizing control feedback.

1 Introduction

In many engineering applications, the mixing of two or more fluids is essential to obtaining good performance in some downstream process (a prime example is the mixing of air and fuel in combustion engines [11, 2]). As a consequence, mixing has been the focus of much research, but without reaching a unified theory, either for the generation of flows that mix well due to external forcing, or for the quantification of mixing in such flows (see [25] for a review). Approaches range from experimental design and testing to modern applications of dynamical systems theory. The latter was initiated by Aref [4], who studied chaotic advection in the setting of an incompressible, inviscid fluid contained in a (2D) circular domain, and agitated by a point vortex (the blinking vortex flow). Ottino and coworkers studied a number of various flows, examining mixing properties based on dynamical systems techniques [7, 18, 20, 32]. Later Rom-Kedar et al. [30] applied Melnikov's method and KAM (Kolmogorov-Arnold-Moser) theory to quantify transport in a flow governed by an oscillating vortex pair. For a general treatment of dynamical systems theory, see, for instance, [12], and for background material related to transport in dynamical systems, see [34]. An obvious shortcoming of this theory is the requirement that the flow must be periodic, as such methods rely on the existence of a Poincaré map for which some periodic orbit of the flow induces a hyperbolic fixed point. Another shortcoming is that they can only handle small perturbations from integrability, whereas effective mixing usually occurs for large perturbations [26]. A third shortcoming is that traditional dynamical systems theory is concerned with asymptotic, or long-time, behavior, rather than quantifying rate processes which are of interest in mixing applications. In order to overcome some of these shortcomings, recent advances in dynamical systems theory have focused on finding coherent structures and invariant manifolds in experimental datasets, which are finite in time and generally aperiodic. This has led to the notions of finite-time hyperbolic trajectories with corresponding finite-time stable and unstable manifolds [13, 14]. The results include estimates for the transport of initial conditions across the boundaries of coherent structures. In [29] these concepts were applied to a time-dependent velocity field generated by a double-gyre ocean model, in order to study the fluid transport between dynamic eddies and a jet stream. An application to meandering jets was described in [24]. Another method for identifying regions in a flow that have similar finite-time statistical properties based on ergodic theory was developed and applied in [22, 21, 23]. The relationship between the two methods mentioned, focusing on geometrical and statistical properties of particle motion, respectively, was examined in [28].

As these developments have partly been motivated by applications in geophysical flows, they are diagnostic in nature and lend little help to the problem of *generating* a fluid flow that mixes well. The problem of generating effective mixing in a fluid flow is usually approached by trial and error using various "brute force" open-loop controls, such as mechanical stirring or jet injection. However, in the recent papers [8, 9], control systems theory was used to rigorously derive the mixing protocol that maximizes entropy among all the possible periodic sequences composed of two shear flows orthogonal to each other. The shear flows are realizable by dragging a plate over the fluid.

In this paper, we propose using active feedback control in order to enhance existing instability mechanisms in a 2D model of plane channel flow. The fluid is considered incompressible and Newtonian (constant viscosity). Our hypothesis is that effective mixing may be obtained by enhancing the instability of the parabolic profile of the Poiseuille flow using boundary control. Furthermore, it is expected that by applying boundary

[1] This work was supported by the Norwegian Research Council, the Office of Naval Research, the Air Force Office of Scientific Research, and the National Science Foundation.

[2] Deparment of Applied Mechanics and Engineering Sciences, University of California, San Diego, CA 92093-0411.

[3] Department of Engineering Cybernetics, Norwegian University of Science and Technology, N-7491 Trondheim.

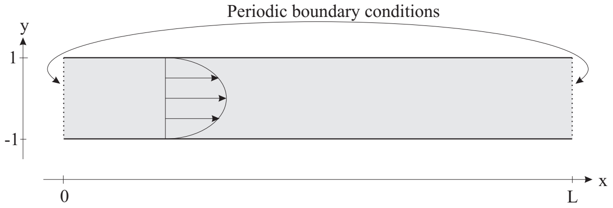

Figure 1: Geometry of the flow problem.

control intelligently in a feedback loop, mixing will be considerably enhanced with relatively small control effort. We design a decentralized control law based on Lyapunov stability analysis, show that it has a significant stabilizing effect on the 2D flow, and finally, we switch the sign of the feedback gain to obtain a destabilizing control algorithm.

It is recognized that channel flow instability mechanisms are inherently 3D. Efforts that study the stabilization problem only in 2D are thus inconclusive about physical flows, for which 3D effects are quite significant. When studying the problem of destabilization, however, the situation is markedly different. In this case, studying the 2D problem, rather than being inconclusive about physical flows, is indeed conservative: the neglected 3D instability mechanisms may be expected to substantially increase the rate of mixing beyond that seen in the 2D model presented here. Thus, the study of 2D flow destabilization has important consequences for physical, 3D flows.

2 Problem Statement

The dimensionless Navier-Stokes equations for incompressible flow between two walls are given by

$$\begin{array}{c} \mathbf{W}_t - \frac{1}{R}\Delta\mathbf{W} + (\mathbf{W}\cdot\nabla)\mathbf{W} + \nabla P = 0 \\ div\mathbf{W} = 0 \end{array} \tag{1}$$

where $\mathbf{W} = \mathbf{W}(x,y,t) = (U(x,y,t), V(x,y,t))^T$ is the velocity at location (x,y) and time t, $P = P(x,y,t)$ is the pressure at location (x,y) and time t, and R is the Reynolds number. $(x,y) \in [0,L) \times [-1,1]$ and $t > 0$. Equation (1) has a steady solution, or fixed point $(\bar{U}, \bar{V})$, given as

$$\bar{U}(y) = 1 - y^2 \tag{2}$$

$$\bar{V} = 0 \tag{3}$$

with pressure $\bar{P} = -2x/R$. The geometry of the problem is illustrated in Figure 1, along with the parabolic equilibrium profile. The stability characteristics of $(\bar{U}, \bar{V})$ vary with the Reynolds number. For $R < 5772$, $(\bar{U}, \bar{V})$ is linearly stable (see for instance [27]), that is, infinitesimal perturbations from the parabolic profile will be damped out. For $R > 5772$, $(\bar{U}, \bar{V})$ is unstable. Our main objective in this paper is to enhance mixing in the channel flow. Towards that end, we first present a control law that is analytically proved to be stabilizing for small Reynolds numbers, and show by simulations that it stabilizes $(\bar{U}, \bar{V})$ for large Reynolds numbers. Then, we reverse the control gain to destabilize the flow and thereby enhance mixing.

Defining the error $\mathbf{w} = (u,v) = (U - \bar{U}, V)$, and defining $p = P - \bar{P}$, we get the following set of equations for the error

$$\begin{array}{l} u_t = \frac{1}{R}(u_{xx} + u_{yy}) - uu_x - \bar{U}u_x - vu_y - v\bar{U}' - p_x \\ v_t = \frac{1}{R}(v_{xx} + v_{yy}) - uv_x - \bar{U}v_x - vv_y - p_y \end{array} \tag{4}$$

$$u_x + v_y = 0 \tag{5}$$

for $(x,y) \in [0,L) \times [-1,1]$ and $t > 0$, and with initial conditions

$$\begin{aligned} u(x,y,0) &= u_0(x,y) \\ v(x,y,0) &= v_0(x,y). \end{aligned}$$

We assume periodic boundary conditions in the streamwise direction, that is, we equate the quantities $\mathbf{w}$ and p at $x = 0$ and $x = L$. The boundary conditions on the walls, $y = \pm 1$, are given by the wall-normal boundary control

$$u(x,-1,t) = u(x,1,t) = 0 \tag{6}$$

$$\begin{aligned} v(x,-1,t) &= v(x,1,t) \\ &= k_v\,(p(x,1,t) - p(x,-1,t)) \end{aligned} \tag{7}$$

which is designed for small Reynolds numbers (see [1]) using the Lyapunov function

$$E(\mathbf{w}) = \|\mathbf{w}\|_{L_2}^2 = \int_{-1}^{1}\int_{0}^{L} (u^2 + v^2)\, dx dy. \tag{8}$$

It is worth noting that this control law is of the Jurdjevic-Quinn [17] type (with respect to the Lyapunov function $E(\mathbf{w})$). This endows the control law with inverse optimality with respect to a meaningful cost functional (which is in this case complicated to write).

3 Numerical Demonstration

The main results of this section are that (1) the stabilizing control law stabilizes the 2D unsteady flow model for high values of Reynolds number, (2) the destabilizing control law achieves excellent mixing in the 2D flow model using small amounts of control effort. The reader is reminded of the comments made in the introduction about the conservative nature of the present 2D mixing results in light of the destabilizing 3D effects present in real channel flows at high values of the Reynolds number.

3.1 The computational scheme

The simulations are performed using a hybrid Fourier pseudospectral-finite difference discretization and the fractional step technique based on a hybrid Runge-Kutta/Crank-Nicolson time discretization using the numerical method of [6]. This method is particularly well suited even for the cases with wall-normal actuation because of its implicit treatment of the wall-normal convective terms. The wall-paralel direction is discretized using 128 Fourier-modes, while the wall-normal direction is discretized using energy-conserving central finite differences on a stretched staggered grid with 100 gridpoints. The gridpoints have hyperbolic tangent distribution in the wall-normal direction in order to adequately resolve the high-shear regions near the walls. A fixed flow-rate formulation is used, rather than fixed average pressure gradient, since observations suggest that the approach to equilibrium is faster in this case [16]. The difference between the two formulations is discussed briefly in [31]. The time step is in the range $0.05 - 0.07$ for all simulations.

3.2 Stabilization

Being valid for small Reynolds numbers only, the theoretical results in [1] are of limited practical value. However, they do suggest controller structures worth testing on flows having higher Reynolds numbers. Here, we demonstrate the stabilizing cabability of the control law for flows at $R = 7500$ and $L = 4\pi$. In addition to reporting the time evolution of the energy, $E(\mathbf{w})$, we also consider the (instantaneous) control effort as a measure of performance. The control effort is defined as

$$C(\mathbf{w}) = \sqrt{\int_0^L \left(|\mathbf{w}(x,-1,t)|^2 + |\mathbf{w}(x,1,t)|^2 \right) dx}. \quad (9)$$

A total of three simulations are reported here: $k_v \in [-0.125, -0.08, -0.05]$. As already mentioned, the parabolic equilibrium profile is unstable for $R = 7500$, so infinitesimal disturbances will grow, but the flow eventually reaches a statistically steady state, which we call *fully established flow.* For all simulations, the fully established flow, for which $E(\mathbf{w}) \approx 1.3$, is chosen as the initial data. The vorticity map for the fully established (uncontrolled) flow, is similar to vorticity maps presented in [16], and clearly shows the ejection of vorticity from the walls into the core of the channel as described in [16].

Figure 2 summarizes the results. It is clear that stabilization is obtained in terms of the energy $E(\mathbf{w})$. The ratio of the peak kinetic energy of the control flow, versus the perturbation kinetic energy in the uncontrolled case (drained out by the control), $C(\mathbf{w})^2/E(\mathbf{w})$, is less than 0.25%.

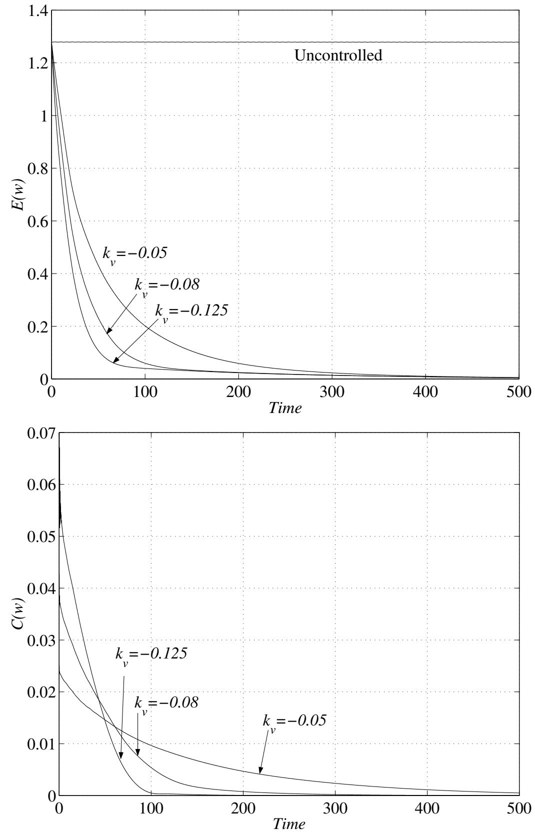

Figure 2: Energy $E(\mathbf{w})$ and control effort $C(\mathbf{w})$ for wall-normal, stabilizing control.

3.3 Mixing

Mixing is commonly induced by means of open loop methods such as mechanical stirring or jet injection. These methods may use excessive amounts of energy, which in certain cases is undesirable. Thus, we propose using active feedback control in order to exploit the natural tendency in the flow to mix. To the authors' knowledge, this is the first attempt to induce mixing by means of feedback, as the mixing protocols thus far have been open loop controls. It was observed in [15] that some heuristic control strategies enhance turbulence, although this observation was not made in the context of mixing but in the context of drag mitigation.

The results of the previous section show that the control law (6)–(7) has a significant stabilizing influence on the 2D channel flow. In this section, we explore the behaviour of the flow when k_v is chosen such that this feedback destabilizes the flow rather than stabilizes it. The conjecture is that the flow will develop a complicated pattern in which mixing will occur. 2D simulations are performed at $R = 6000$, for which the parabolic equilibrium profile is unstable. The initial data for the simulations is the fully established flow. Some mixing might be expected in this flow, as it periodically ejects vorticity into the core of the channel. Our objective, however, is to enhance the mixing process by boundary control, which we impose by setting $k_v = 0.1$ in (7). The upper graph in Figure 3 show the perturbation energy, $E(\mathbf{w})$, which increases by a factor of 5. It is interesting to notice that the control leading to this increase in perturbation energy is small (see middle graph in Figure 3). The maximum value of the control flow kinetic energy is less than 0.7% of the perturbation kinetic energy of the uncontrolled flow, and only about 0.1% of the fully developed, mixed (controlled) flow! Next, we will quantify the mixing in a more rigorous way, and compare the controlled and uncontrolled cases.

A number of inherently different processes constitute what is called mixing. Ottino [25] distinguishes between three sub-problems of mixing: (i) mixing of a single fluid (or similiar fluids) governed by the stretching and folding of material elements; (ii) mixing governed by diffusion or chemical reactions; and (iii) mixing of different fluids governed by the breakup and coalescence of material elements. Of course, all processes may be present simultaneously. In the first sub-problem, the interfaces between the fluids are passive [3], and the mixing may be determined by studying the movement of a passive tracer, or dye, in a homogeneous fluid flow. This is the problem we are interested in here.

The location of the dye as a function of time completely describes the mixing, but in a flow that mixes well, the length of the interface between the dye and the fluid increases exponentially with time. Thus, calculating the location of the dye for large times is not feasible within the restrictions of modest computer resources [10]. We do, nevertheless, attempt this for small times, and supplement the results with less accurate, but computationally feasible, calculations for larger times. A particle-line method, loosely based on [33] and [19], is used to track the dye interface. In short, this method represents the interface as a number of particles connected by straight lines. The positions of the particles are governed by the equation $dX/dt = (U(X,t), V(X,t))$, where X is a vector of particle positions. At the beginning of each time step, new particles are added such that at the end of the time step, a prescribed resolution, given in terms of the maximum length between neighboring particles, is maintained. The fact that we are working with a single fluid representing multiple miscible fluids, ensures that dye surfaces remain connected [26]. At $t = 50$, when the perturbation energy is about tripled in the controlled case (Figure 3), eighteen blobs are distributed along the centerline of the channel as shown in Figure 4. They cover 25% of the total domain. Figure 5 shows the configuration of the dye in the controlled case for 4 time instances. The difference in complexity between the uncontrolled and controlled cases is clear (compare the lower graphs of Figures 4 and 5), however, large regions are poorly mixed even at $t = 85$. The lower graph in Figure 3 shows the total length of the surface of the dye. The length appears to grow linearly with time in the uncontrolled case, whereas for the controlled case, it grows much faster, reaching values an order of magnitude larger than in the uncontrolled case. In order to approximate the dye distribution for large time, a fixed number of particles are uniformly distributed throughout the domain, distinguishing between particles placed on the inside (*black* particles) and on the outside (*white* particles) of regions occupied by dye. Figure 6 shows the distribution of black particles at $t = 85$ (for comparison with Figure 5), 100, 125 and 150. The particle distribution becomes increasingly uniform.

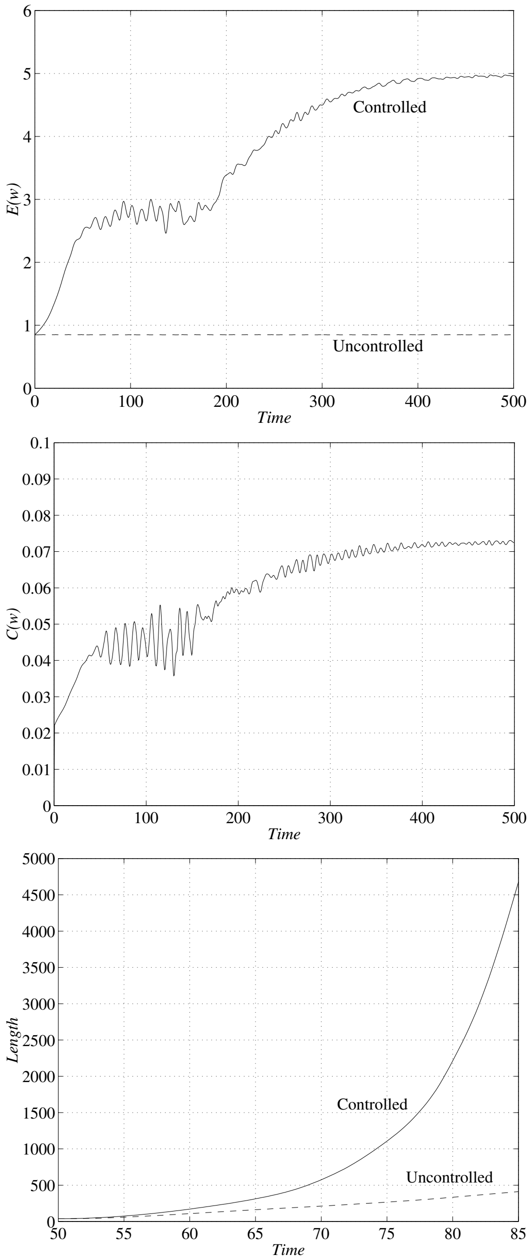

Figure 3: Energy $E(\mathbf{w})$, control effort $C(\mathbf{w})$, and dye surface length, as functions of time.

In order to quantify the mixing further, we ask the

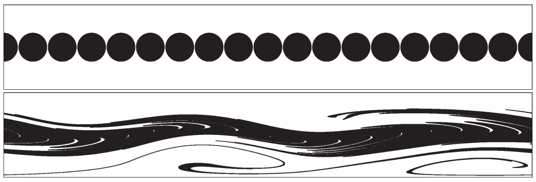

Figure 4: Initial distribution of dye blobs (at $t = 50$), and dye distribution at $t = 85$ for uncontrolled flow.

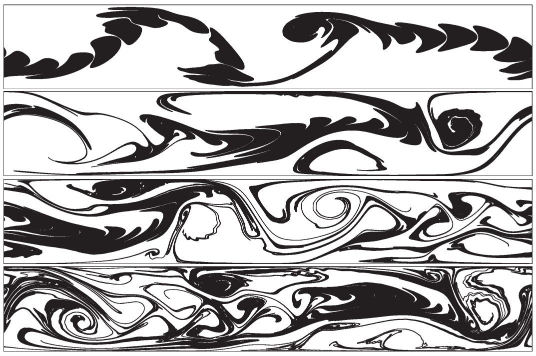

Figure 5: Dye distribution for controlled flow at $t = 55$, 65, 75 and 85 (from top towards bottom).

following question: given a box of size ε, what is the probability, P, of the fluid inside being *well mixed*? An appropriate choice of ε, and what is considered well mixed, are application specific parameters, and are usually given by requirements of some downstream process. In our case, the blobs initially cover 25% of the domain, so we will define *well mixed* to mean that the dye covers between 20% and 30% of the area of the box. The size ε of the boxes will be given in terms of pixels along one side of the box, so that the box covers ε^2 pixels out of a total of 2415×419 pixels for the entire domain. On this canvas, the box may be placed in $(419 - (\varepsilon - 1)) \times 2415$ different locations. The fraction of area covered by dye inside box i of size ε, is for small times calculated according to

$$c_\varepsilon^i = \frac{n_p}{\varepsilon^2} \tag{10}$$

where n_p is the number of pixels covered by dye, and for large times according to

$$c_\varepsilon^i = \frac{n_b}{n_w + n_b} \tag{11}$$

where n_b and n_w denote the number of black and white particles, respectively, contained in the box. P , which depends on ε, is calculated as follows

$$P_\varepsilon = \frac{1}{n} \sum_{i=1}^{n} eval(0.2 < c_\varepsilon^i < 0.3) \tag{12}$$

where n is the total number of boxes. The expression in the summation evaluates to 1 when $0.2 < c_\varepsilon^i < 0.3$ and 0 otherwise. For small times $n = (419 - (\varepsilon - 1)) \times$

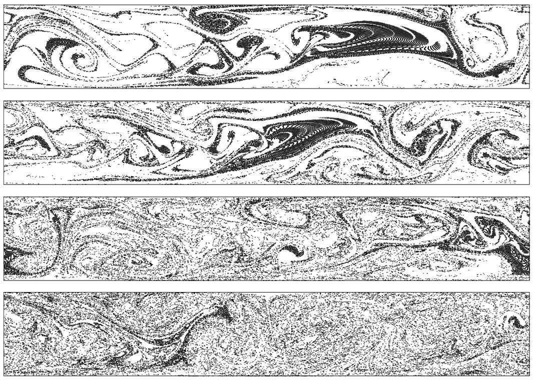

Figure 6: Particle distribution for controlled flow at $t = 85$, 100, 125 and 150 (from top towards bottom).

2415, whereas for large times n may be smaller as we choose to ignore boxes containing less than 25 particles. Figures 7 and 8 show P_ε as a function of time for $\varepsilon \in [15, 30, 45, 60]$.

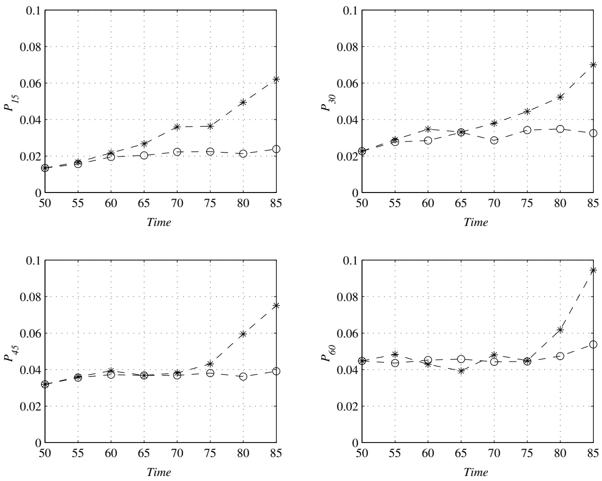

Figure 7: Probability of well mixedness for the uncontrolled case (o) and controlled case (*).

4 Conclusions

We have addressed the problem of imposing mixing by means of boundary feedback control in 2D channel flow. This is done by first designing a control law for the stabilization of the parabolic equilibrium profile in 2D, and then reversing the sign of the feedback gain in order to obtain destabilization. The result is a highly complex flow pattern and improved mixing, as confirmed by studies of the behaviour of dye blobs positioned in the flow. The mixing is obtained by a small control effort, compared to the reference velocity of the flow. This is the main advantage of applying control intelligently in a feedback loop.

Acknowledgements

We thank Peter Blossey for providing the channel flow computer code used for the simulations, and for

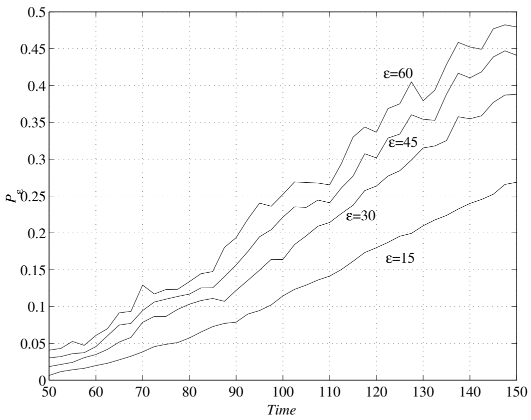

Figure 8: Probability of well mixedness for the controlled case based on uniform particle distribution.

his helpful comments while adapting it to the two-dimensional boundary control problem. We also thank Andras Balogh for various discussions.

References

[1] O.M. Aamo, M. Krstić and T.R. Bewley. Control of Mixing by Boundary Feedback in 2D Channel Flow. Submitted.

[2] A.M. Annaswamy and A.F. Ghoniem. Active Control in Combustion Systems. IEEE Cont. Sys., vol. 15, no. 6, pp. 49-63, 1995.

[3] H. Aref and G. Tryggvason. Vortex dynamics of passive and active interfaces. Physica, 12D, 59-70, 1984.

[4] H. Aref. Stirring by chaotic advection. J. Fluid Mech., vol. 143, pp. 1-21, 1984.

[5] A. Balogh, W-J. Liu and M. Krstić. Stability Enhancement by Boundary Control in 2D Channel Flow. Accepted subject to revision for IEEE Trans. Automat. Contr., 2000.

[6] T.R. Bewley and P. Moin. Technical report TF-76, Stanford University, 1999.

[7] W.-L. Chien, H. Rising and J.M. Ottino. Laminar mixing and chaotic mixing in several cavity flows. J. Fluid Mech., vol. 170, pp. 355-377, 1986.

[8] D. D'Alessandro, M. Dahleh and I. Mezić. Control of Fluid Mixing Using Entropy Methods. In Proc. Am. Contr. Conf., Philadelphia, PA, 1998.

[9] D. D'Alessandro, M. Dahleh and I. Mezić. Control of Mixing in Fluid Flow: A Maximum Entropy Approach. IEEE Trans. Automat. Contr., vol. 44, no. 10, pp.1852-1863, 1999.

[10] J.G. Franjione and J.M. Ottino. Feasibility of numerical tracking of material lines and surfaces in chaotic flows. Phys. Fluids, vol. 30, pp. 3641-3643, 1987.

[11] A.F. Ghoniem and K.K. Ng. Numerical study of the dynamics of a forced shear layer. Phys. Fluids, vol. 30, no. 3, pp. 706-721, 1987.

[12] J. Guckenheimer and P. Holmes. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer-Verlag, 1983.

[13] G. Haller and A.C. Poje. Finite time transport in aperiodic flows. Physica D, 119, pp. 352-380, 1998.

[14] G. Haller. Finding finite-time invariant manifolds in two-dimensional velocity fields. Chaos, vol. 10, no. 1, 2000.

[15] E.P. Hammond, T.R. Bewley and P. Moin. Observed mechanisms for turbulence attenuation and enhancement in opposition-controlled wall-bounded flows. Phys. Fluids, vol. 10, no. 9, pp. 2421-2423, 1998.

[16] J. Jiménez. Transition to turbulence in two-dimensional Poiseuille flow. J. Fluid Mech., vol. 218, pp. 265-297, 1990.

[17] V. Jurdjevic and J.P. Quinn. Controllability and Stability. J. Diff. Eq., vol. 28, pp. 381-389, 1978.

[18] D.V. Khakhar, H. Rising and J.M. Ottino. Analysis of chaotic mixing in two model systems. J. Fluid Mech., vol. 172, pp. 419-451, 1986.

[19] T.S. Krasnopolskaya, V.V. Meleshko, G.W.M. Peters and H.E.H Meijer. Mixing in Stokes flow in an annular wedge cavity. Eur. J. Mech. B/Fluids, 18, pp. 793-822, 1999.

[20] C.W. Leong and J.M. Ottino. Experiments on mixing due to chaotic advection in a cavity. J. Fluid Mech., vol. 209, pp. 463-499, 1989.

[21] N. Malhotra, I. Mezić and S. Wiggins. Patchiness: A new diagnostic for Lagrangian trajectory analysis in time-dependent fluid flows. Int. J. Bifurc. and Chaos, vol. 8, no. 6, pp. 1053-1093, 1998.

[22] I. Mezić. On geometrical and statistical properties of dynamical systems: Theory and applications. Ph.D. thesis, California Institute of Technology, 1994.

[23] I. Mezić and S. Wiggins. A method for visualization of invariant sets of dynamical systems based on the ergodic partition. Chaos, vol. 9, no. 1, pp. 213-218, 1999.

[24] P.D. Miller, C.K.R.T. Jones, A.M. Rogerson and L.J. Pratt. Quantifying transport in numerically generated velocity fields. Physica D, 110, pp. 105-122, 1997.

[25] J.M. Ottino. The kinematics of mixing: stretching, chaos, and transport. Cambridge University Press, 1989.

[26] J.M. Ottino. Mixing, chaotic advection, and turbulence. Annu. Rev. Fluid Mech., 22: 207-53, 1990.

[27] R.L. Panton. Incompressible flow. Second Edition. John Wiley & Sons, Inc., 1996.

[28] A.C. Poje, G. Haller and I. Mezić. The geometry and statistics of mixing in aperiodic flows. Phys. Fluids, vol. 11, no. 10, 1999.

[29] A.C. Poje, G. Haller. Geometry of Cross-Stream Mixing in a Double-Gyre Ocean Model. J. of Phys. Ocean., vol. 29, pp. 1469-1665, 1999.

[30] V. Rom-Kedar, A. Leonard and S. Wiggins. An analytical study of transport, mixing and chaos in an unsteady vortical flow. J. Fluid Mech., vol. 214, pp. 347-394, 1990.

[31] B.L. Rozhdestvensky and I.N. Simakin. Secondary flows in a plane channel: their relationship and comparison with turbulent flows. J. Fluid Mech., vol. 147, pp. 261-289, 1984.

[32] P.D. Swanson and J.M. Ottino. A comparative computational and experimental study of chaotic mixing of viscous fluids. J. Fluid Mech., vol. 213, pp. 227-249, 1990.

[33] A.A. Ten, Y.Y. Podladchikov, D.A. Yuen, T.B. Larsen and A.V. Malevsky. Comparison of Mixing Properties in Convection with the Particle-Line Method. Geophys. Res. Lett. 25, 3205-3208, 1998.

[34] S. Wiggins. Chaotic Transport in Dynamical Systems. Springer-Verlag, 1992.