A simple method for constructing Lyapunov functions to prove convergence of exponentially stabilized second-order oscillators

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Abstract

Bewley (2022) considered the control of a variable-length pendulum with a simple nonlinear feedback strategy. Though Lyapunov stability of the resulting controlled system was demonstrated in the discrete-time sense of Karafyllis (2012), with a candidate Lyapunov function $V_s(t)$, as it evolved along controlled system trajectories, reducing by a certain fraction after every 2π radians of rotation of the system trajectory around the origin in phase space, the candidate Lyapunov function $V_s(t)$ proposed did not decrease montonically. The present paper shows how a nonmonotonic candidate Lyapunov function $V_s(t)$ in such a setting can easily be modified to generate a new function with $V(t) > 0$ and $\dot{V}(t) < 0$ along system trajectories in a finite region around the origin, thus establishing exponential stability in the classic sense of Lyapunov.

1 Introduction

The nondimensionalized variable-length pendulum, aka the "varpend oscillator", is governed (see, e.g., [1]) by

$$\ddot{\phi} = -(\sin\phi + 2\,\dot{\ell}\,\dot{\phi})/\ell, \tag{1}$$

where $\phi(t)$ denotes the angle of the pendulum, $\ell(t) = L(t)/L_0$ is the normalized length of the pendulum, L_0 is the nominal length of the pendulum, $t = \tau/\tau_0$ is the nondimensionalized time variable, τ denotes the original (dimensional) time variable, and $\tau_0 = \sqrt{L_0/g}$ is the characteristic (dimensional) time constant, where g is the effective acceleration due to gravity. As studied in [2], simple nonlinear feedback of the form

$$\ell(t) = 1 + \delta(t)\,\phi(t)\,\dot{\phi}(t), \tag{2a}$$
$$\delta(t) = C/V_s(t), \tag{2b}$$
$$V_s(t) = \dot{\phi}(t)^2/2 + 1 - \cos\phi(t) \tag{2c}$$

exponentially stabilizes this system. It was further shown in [2] that, combining (1) and (2), the second-order ODE governing this system may be written

$$\ddot{\phi} + \sin\phi = \frac{-(2\,C\,\dot{\phi}^3/V_s)\,[1 - \phi\,(\sin\phi)/V_s] + [3\,C\,\phi\,\dot{\phi}/V_s - 2\,C\,\phi\,\dot{\phi}^3/V_s^2]\,\sin\phi}{1 + 3\,C\,\phi\,\dot{\phi}/V_s - 2\,C\,\phi\,\dot{\phi}^3/V_s^2}, \tag{3}$$

where $V_s(t)$ is a simple function of $\phi(t)$ and $\dot{\phi}(t)$ [see (2c)], and that we may ensure that the denominator on the RHS of (3) remains positive, and thus that $\ddot{\phi}$ remains bounded, by assuming that

$$0 < C < C_{\max} = [(69 - 11\sqrt{33})/2]^{1/2}/3 \approx 0.56813. \tag{A2}$$

This upper bound on the maximum allowable C was shown to be tight; for $C = C_{\max}$, for a certain value of $\dot{\phi}/\phi$, the denominator of (3) goes to zero, and $\ddot{\phi}$ diverges. For $C < C_{\max}$, no combinations of ϕ and $\dot{\phi}$ drive the denominator to zero. In practice, C should be kept significantly smaller than $C_{\max}$ in order to ensure a well-behaved controlled system, accurately governed by the model given in (1) with appropriately small accelerations $\ddot{\phi}$.

The dynamics of this controlled system, taking $C = 0.1$, is demonstrated in Figure 1 via numerical simulation (using standard RK4 with $\Delta t = 0.01$, and ICs of $\phi(0) = 1$ and $\phi'(0) = 0$); exponential convergence is observed over 13 orders of magnitude, down to machine zero.

Though the system illustrated in Figure 1 is clearly exponentially stable, the fact that $V_s(t)$ did not reduce monotonically was particularly annoying.

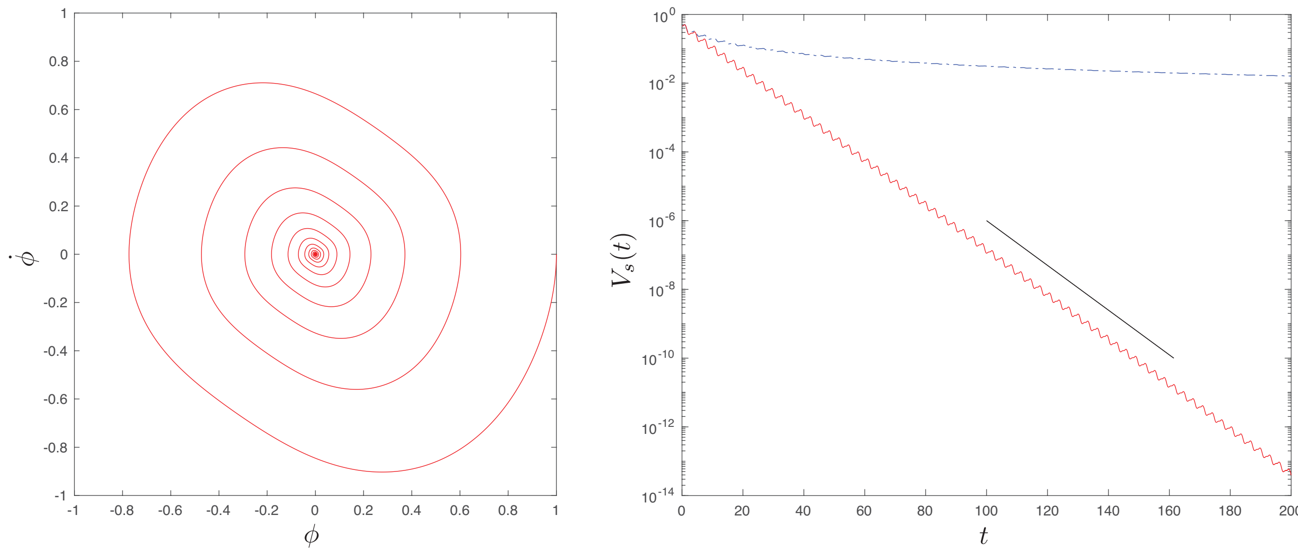

Figure 1: Simulation of the exponentially-stabilized varpend oscillator (1) with (2) [equivalently, (3)], taking $C = 0.1$. (a) Trajectory in phase space $\{\phi, \dot{\phi}\}$, and (b) a simplified measure of the energy of the oscillations of $\phi(t)$ as a function of time, $V_s(t)$. For reference, the black line in (b) indicates a slope corresponding to a reduction of $V_s(t)$ by an order of magnitude every $\Delta t = 2.445 \cdot 2\pi$ nondimensional time units.

References

[1] Wirkus, S., Rand, R., & Ruina, A. (1998) How to Pump a Swing, *Coll. Math. J* **29** (4) 266-275.

[2] Bewley, T (2022) Exponential stabilization of a variable-length pendulum with nonlinear feedback and a curious caveat. *Automatica*, submitted.